Minimal Public Communication for Maximum Rate Secret Key Generation

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Secret Key Generation

Secret key (SK) $K$ with interactive communication $F$ satisfies:

$$\Pr(K_1 = K_2 = K) \approx 1 : \text{ Recoverability}$$

$$\frac{1}{n} I(K \wedge F) \approx 0 : \text{ Secrecy}$$

Rate of the secret key $= \frac{1}{n} H(K)$. 

Secret key capacity $C = \text{ maximum achievable rate of a secret key}$.  

[Maurer ‘93, Ahlswede-Csiszár ‘93]

$$C = I(X \wedge Y).$$
What is the min. rate of $F$ required for achieving SK capacity?

- **Maurer-Ahlswede-Csiszár**
  - *Common randomness* (CR) generated: $X^n$ or $Y^n$.
  - Rate of communication required $= \min\{H(X|Y); H(Y|X)\}$.
  - Decomposition:
    \[
    H(X) = H(X|Y) + I(X \land Y), \\
    H(Y) = H(Y|X) + I(X \land Y).
    \]

- **Csiszár-Narayan**
  - *Common randomness* generated: $X^n, Y^n$.
  - Rate of communication required $= H(X|Y) + H(Y|X)$.
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    H(X,Y) = H(X|Y) + H(Y|X) + I(X \land Y).
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Communication for SK Capacity

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Q1: Forms of CR for agreeing on optimum rate SK?

Q2: Does minimum communication rate correspond to “minimum” CR?
Wyner’s Common Infomation

\[ CI(X \land Y) \equiv \text{min. rate of a function } L = L(X^n, Y^n) \text{ such that} \]

\[ \frac{1}{n} I(X^n \land Y^n \mid L) \approx 0. \]

Defined in the context of source generation and source coding.
Interactive Form of Wyner’s Common Information

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  Defined in the context of source generation and source coding.

- Interactive Common Information
  Terminals agree on CR \( J \) using \( r \)-rounds \( F \).
  \[ CI_i^r(X \land Y) \equiv \min \text{ rate of } (J, F) \text{ such that } \]
  \[ \frac{1}{n} I(X^n \land Y^n \mid J, F) \approx 0. \]

  \[ CI_i(X \land Y) = \lim_{r \to \infty} CI_i^r(X \land Y). \]

  Note: \( CI(X \land Y) \leq CI_i(X \land Y) \leq \min \{H(X); H(Y)\}. \)
Minimum Communication for Optimum SK

For minimum rate of communication $R_{SK}$ for optimum rate SK:

We characterize the CR associated with optimum rate SK.

- A CR $J$ recoverable from communication $F \Rightarrow$ SK of rate $\frac{1}{n} H(J|F)$.

- An optimum rate SK corresponds to a CR $J$ recoverable from $F$ s.t.

\[ \frac{1}{n} I(X^n \land Y^n \mid J, F) \approx 0. \]

- For instance: $J = X^n, Y^n$ or $(X^n, Y^n)$.

- $R_{CI}$ be the min. rate of communication for such CR.

- Shall show: CR of the above form always yields optimum rate SK.
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  - For instance: $J = X^n$, $Y^n$ or $(X^n, Y^n)$.

  - $R_{CI}$ be the min. rate of communication for such CR.

- Shall show: CR of the above form always yields optimum rate SK.

Theorem

\begin{equation}
R_{SK} = R_{CI} = CI_i(X \land Y) - I(X \land Y).
\end{equation}
Minimum Communication for Optimum SK

**Theorem**

\[ R_{SK} = R_{CI} = CI_i(X \land Y) - I(X \land Y). \]

- **Lemma:** For each \( r \geq 1 \):
  - \( R^r_{SK} \geq R^r_{CI} \geq R^{r+1}_{SK} \).
  - Decomposition:
    \[ R^r_{CI} \geq CI^r_i(X \land Y) - I(X \land Y) \geq R^{r+1}_{CI}. \]
- Theorem follows by taking the limit \( r \to \infty \) in the Lemma.
Idea of the Proof

Proof is based on the observation:

\[ I(X \land Y) \approx \frac{1}{n} [I(X^n \land Y^n \mid J, F) + H(J, F) - H(F \mid X^n) - H(F \mid Y^n)]. \]

Characterization of the form of CR in optimum SK generation:

\[ \frac{1}{n} I(X^n \land Y^n \mid J, F) \approx 0 \]

if and only if

\[ I(X \land Y) \approx \frac{1}{n} \left[ H(J, F) - [H(F \mid X^n) + H(F \mid Y^n)] \right]. \]
Proof is based on the observation:

\[ I(X \land Y) \approx \frac{1}{n} \left[ I(X^n \land Y^n | J, F) + H(J, F) - H(F | X^n) - H(F | Y^n) \right]. \]

Characterization of the form of CR in optimum SK generation:

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SK rate associated with CR $J$ recoverable from $F$
Minimum Communication for Optimum SK

**Theorem**

\[ R_{SK} = \text{min. rate of } F \text{ required for optimal rate SK generation.} \]
\[ R_{CI} = \text{min. rate } F \text{ required for generating CR } J \text{ s.t. } X^n \, \bigtriangleup \, Y^n | (J, F). \]

Then,

\[ R_{SK} = R_{CI} = CI_i(X \wedge Y) - I(X \wedge Y). \]

\[ CI_i(X \wedge Y) = \lim_{r \to \infty} CI_i^r(X \wedge Y). \]
Characterization of $CI_i$

- Given rvs $X, Y$ and $r \geq 1$, we have

$$CI^r_i(X \land Y) = \min_{U_1, V_1, \ldots, U_r, V_r} I(X, Y \land U_1, V_1, \ldots, U_r, V_r),$$

- \[ \begin{align*}
U_1 & \leftarrow\rightarrow X \leftarrow\rightarrow Y, \\
U_2 & \leftarrow\rightarrow X, V_1 \leftarrow\rightarrow Y, \\
& \vdots \\
U_r & \leftarrow X, U^{r-1}, V^{r-1} \leftarrow Y,
\end{align*} \]

- \[ \begin{align*}
V_1 & \leftarrow\rightarrow Y, U_1 \leftarrow\rightarrow X, \\
V_2 & \leftarrow\rightarrow Y, U_1, V_1 \leftarrow\rightarrow X, \\
& \vdots \\
V_r & \leftarrow Y, U^r, V^{r-1} \leftarrow X.
\end{align*} \]

- $X \leftarrow\rightarrow U^r, V^r \leftarrow\rightarrow Y$

- Single-letter expression for $CI_i$ is not (yet) available.
Noninteractive Communication for SK Capacity

Communication from $X^n$ to $Y^n$:

$$R_{SK}^{one} = \min_{U \leftarrow X \rightarrow Y, X \leftarrow U \rightarrow Y} I(X, Y \wedge U) - I(X \wedge Y)$$

$U$ satisfies: $U \leftarrow X \rightarrow Y$, $X \leftarrow U \rightarrow Y$.

▶ [Problem 16.26, Csiszár-Körner]

$$\min_{U \leftarrow X \rightarrow Y, X \leftarrow U \rightarrow Y} I(X, Y \wedge U) = \min_{g(X) \text{ s.t. } X \leftarrow g(X) \rightarrow Y} H(g(X))$$
Common Information Quantities

For a pair of rvs $X, Y$

$$CI_{GC} \leq I(X \wedge Y) \leq CI \leq CI_i \leq CI_i^r \leq CI_i^{r-1} \leq \min\{H(X); H(Y)\}$$
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Interactive Common Information
Common Information Quantities

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Interactive Common Information

- $CI_i$ is indeed a new quantity

For binary rvs $X$ and $Y$: $CI_i(X \land Y) = \min\{H(X); H(Y)\}$.

For binary symmetric $X$ and $Y$: $CI(X \land Y) < \min\{H(X); H(Y)\}$. 