Coverage in a Poisson-Boolean Model

- Motivation
- Let \( \Phi \) denote a homogeneous Poisson point process on \( \mathbb{R}^d \) of intensity \( \lambda \).
- \( \Phi(A) = \# \text{ of points falling in the set } A \subseteq \mathbb{R}^d \).
- \( \Phi(A) \sim \text{Poi}(\lambda |A|) \), where \( |A| \) = volume of \( A \).
- \[ P[\Phi(A) = k] = e^{-|A|\lambda} \frac{|A|\lambda^k}{k!}, k = 0, 1, \ldots \]
- \( E[\Phi(A)] = \lambda |A| \)
- \( \lambda = \text{expected } \# \text{ of points falling in a unit volume} \).
- If \( A, B \subseteq \mathbb{R}^d \) are disjoint, then \( \Phi(A) \) and \( \Phi(B) \) are independent random variables.
- Let \( \Phi = \{X_1, X_2, \ldots\} \)
- \( B(x, r) = \text{ball of radius } r \text{ centered at } x \).
- \( |B(x, r)| = \theta \).

Def: The Poisson-Boolean model or the coverage process is defined as:
\[ \mathcal{C} = \mathcal{C}(\lambda, r) = \bigcup_{i=1}^{\infty} B(X_i, r) \]

Def: The vacancy of a smooth bounded region \( R \subseteq \mathbb{R}^d \) is the region within \( R \) that is not covered by \( \mathcal{C} \), i.e.
\[ V(R) = \int_{R^c} \frac{1}{x} \, dx \]
\[ \left[ \begin{array}{l} 1 \quad \text{if } x \in \mathcal{A} \\ 0 \quad \text{otherwise} \end{array} \right] \]
- \( E[V(R)] = E\left[ \int_{R^c} \frac{1}{x} \, dx \right] = \int_{R^c} E\left[ \frac{1}{x} \right] \, dx \)
- \( = \int_{R^c} \int_{R} P[x \notin \mathcal{C}^c] \, dx \)
- \( = \int_{R^c} P[\Phi(B(x, r)) = 0] \, dx \)
- \( = \int_{R^c} e^{-\lambda B(x, r)} \, dx \)
- \[ E[V(R)] = |R| e^{-\lambda \theta^d} \]

Remark: So if we want not more than \( \kappa \) proportion of the region \( R \) to be vacant (not covered) on average, then \( \kappa e^{-\lambda \theta^d} \leq 1 \).
\[
E[V(R)^2] = E \left[ \int_{\mathcal{C}^d} 1_{C^n(x)} \, dx \right]^2 \\
= \left[ \int_{\mathcal{C}^d} \int_{\mathcal{C}^d} 1_{C^n(x)} 1_{C^n(y)} \, dx \, dy \right] \\
= \int_{\mathcal{C}^d} \int_{\mathcal{C}^d} P\{C^n(B(x,n) \cup B(y,n)) = 0\} \, dx \, dy \\
= \int_{\mathcal{C}^d} \int_{\mathcal{C}^d} 1 - 1_{B(x,n) \cap B(y,n)} \, dx \, dy \\
= e^{2\lambda \theta^n d} \int_{\mathcal{C}^d} \left( e^{2\lambda \theta^n d} 1_{B(x,n) \cap B(y,n)} \right) \, dx \, dy. \\
\text{Var}(V(R)) = e^{2\lambda \theta^n d} \int_{\mathcal{C}^d} \left( e^{2\lambda \theta^n d} 1_{B(x,n) \cap B(y,n)} \right) \, dx \, dy. \\
\]

\text{Theoretical Results:}

1) Strong Law: If \( \lambda \to \infty \) such that \( \lambda \theta^n d \to s > 0 \), then \( V(R) \to 1R1 e^{-s} \) almost surely.

2) \( \lambda \text{ Var}(V(R)) \to 0 \) as \( \lambda \to \infty \), \( \lambda \theta^n d \to s > 0 \).

\[
\text{CLT} \quad d\bar{\alpha} \left[ V(R) - E[V(R)] \right] \to N(0, \sigma^2). \\
\]

\text{Extensions:}

1) Cox process
2) Poisson cluster process
3) k-coverage
4) On-off process

\text{Complete Coverage}

Q: How should \( \lambda \) scale with \( \lambda \) as \( \lambda \to \infty \) so that \( R \) is completely covered with probability approaching one?
**II. PERCOLATION**

Percolation refers to the existence of an infinite/giant component in a graph.

1. **Bond percolation in \( \mathbb{Z}^2 \)** (the random grid)

Graph \( G_p \) with vertex set \( V = \mathbb{Z}^2 \) & edge set

\[ E_p = \{ \langle x, y \rangle : x, y \in \mathbb{Z}^2, |x - y| = 1 \} \]

Each edge is open w.r.t. \( p \) & closed w.r.t. \( 1-p \), independent of other edges.

This yields the random graph \( G_p \)

\[ E_p = \{ \langle x, y \rangle \in E : \langle x, y \rangle \text{ is open} \} \]

\[ G_p = (\mathbb{Z}^2, E_p) \]

**Def.** A **bond path** is said to exist from \( u \in \mathbb{Z}^2 \) to \( v \in \mathbb{Z}^2 \) if

\[ \langle u \rangle \to \langle v \rangle \text{ s.t. } \langle u \rangle = \langle u_0, u_1 \rangle, \langle v \rangle = \langle v_n, v_{n+1} \rangle \in E_p, \]

\[ i = 0, \ldots, n \text{ s.t. } \]

**Def.** A **connected component** is a maximal set of vertices s.t. for any two vertices in the set there is a path from one to the other.

**Def.** The network \( G_p \) is said to **percolate** if it contains an infinite connected component.

\[ \psi(p) = \Pr(\exists \text{ an infinite connected component in } G_p) \]

\[ = \Pr(\text{G}_p \text{ percolates}) \]

**Def.** A phase transition is said to occur at a critical point \( p_c \in (0,1) \) if

\[ \psi(p) = 0 \quad \forall p < p_c \quad \& \quad \psi(p) = 1 \quad \forall p > p_c. \]

**Def.** \( C(x) = \text{connected component containing } x \in \mathbb{Z}^2 \).

\[ C(0) \]

\[ |C(x)| = \text{cardinality of } C(x) \]

\[ \theta(p) = \Pr_C(1 \leq |C(x)| = \infty) \]

- **percolation probability**

**Thm.** If \( 0 < p_1 < p_2 < 1 \) then \( \theta(p_1) \leq \theta(p_2) \).

**Thm. (Kesten 1980)** \( p_c = \frac{1}{2} \)

*Easy to show to \( \frac{1}{2} \leq p_c < \frac{3}{4} \)*

- At most one infinite connected component

\[ \psi(p) = 0 \quad \text{for } p \leq \frac{1}{2} \]

\[ \psi(p) = 1 \quad \text{for } p > \frac{1}{2} \]

**Th.** \( \psi(p) = \Pr_{G_p}(1 \leq |C(0)| = \infty) \)

\[ \leq \sum_{x \in \mathbb{Z}^2} \theta(x) = 0 \quad \text{if } p \leq p_c. \]

- **percolation** of \( G_p \) does not depend on the state of any finite collection of edges.

Kolmogorov's 0-1 Law \( \implies \psi(p) = 0 \) or 1

\[ \psi(p) > 0 \implies \theta(p) > 0 \text{ if } p > p_c \]

\[ \implies \psi(p) = 1 \text{ if } p > p_c. \]

**Implication for finite Graphs**

Let \( G_{n,p} \) be \( G_p \) restricted to edges in \( [0, n]^2 \). Then for large \( n \), if \( p < p_c \) then components are of size \( O(\log n) \). If \( p > p_c \) then largest component is \( O(n) \) & second largest is \( O(\log n) \).
The Random Connection Model

\( \Phi_a \) be a homogeneous Poisson point process on \( \mathbb{R}^2 \) of intensity \( \lambda \).

\( g: \mathbb{R}^2 \rightarrow [0,1] \), \( g(x) \) depends only on \( |x| \) and is non-increasing.

Let \( \forall x, y \in \Phi_a \) be connected by an edge w.p. \( g(|x-y|) \).

Assumption: \( \int_{\mathbb{R}^2} g(x) dx < \infty \)

\( \mathbb{P}^x \) denote the Palm measure, i.e. the distribution of \( \Phi_a \) conditioned to have a point at \( x \).

Distribution of \( \Phi_a \) under \( \mathbb{P}^x \) is same as that of \( \Phi \cup \{x\} \) under \( \mathbb{P} \).

Let \( \tilde{\Phi} \) be the points of \( \Phi_a \) connected by an edge to \( x \) under \( \mathbb{P}^x \).

\( \tilde{\Phi} \) is a non-homogeneous Poisson point process of intensity \( 2g(x) \).

\[ E[\tilde{\Phi}(\mathbb{R}^2)] = \int_{\mathbb{R}^2} g(y) dy \]

The reason for the assumption above.

1. \( C \) = cardinality of component containing origin 0 under \( \mathbb{P}^0 \).

\[ \Theta(\lambda) = \mathbb{P}^0(C = 0) \]

Thm: \( \forall \lambda \in (0, \infty) \), \( \Theta(\lambda) = 0 \) for \( \lambda < \lambda_c \) and \( \Theta(\lambda) > 0 \) for \( \lambda > \lambda_c \).

The Boolean Model

\[ g(x) = \begin{cases} 1 & \text{if } |x| < 2r \\ 0 & \text{otherwise} \end{cases} \]

Average degree of a node (\( \xi \))

\[ = 4\pi r^2 \lambda_c \]

Thm: \( i) \) Fix \( \lambda > 0 \). \( \forall \lambda_c > 0 \) s.t. \( \lambda_c > \lambda \), \( \Theta(\lambda) = 0 \) and \( \Theta(\lambda_c) > 0 \).

\( ii) \) Fix \( \lambda > 0 \). \( \forall \lambda_c > 0 \) s.t. \( \lambda_c > \lambda \), \( \Theta(\lambda) = 0 \) and \( \Theta(\lambda_c) > 0 \).

\( iii) \) Fix \( \lambda > 0 \). \( \forall \lambda_c > 0 \) s.t. \( \lambda_c > \lambda \), \( \Theta(\lambda) = 0 \) and \( \Theta(\lambda_c) > 0 \).

\[ \lambda_c = 4\pi r^2 \lambda_c \approx 4.512 \]

Thm: Fix \( \lambda > 0 \). Let \( \lambda > \lambda_c \). \( R_{dn} \) denote a left-right crossing of a rectangle \( d_n \times d_n \) of sides \( d_n \times d_n \) for \( n \to \infty \). Then \( \mathbb{P}(R_{dn} \uparrow) \to 1 \) as \( n \to \infty \).

\[ \mathit{d_n} \]

\[ \uparrow \]

\[ \downarrow \]

\[ d_n \]

\[ \uparrow \]

\[ \downarrow \]
Almost Connectivity

\( \mathcal{E} \) - Homogeneous PPP(1) on \( \mathbb{R}^2 \)

\( B_n = \left[ 0, \frac{\pi}{\sqrt{n}} \right]^2 \)

\( G_n(\mathcal{E}) = \text{graph with vertex set } \mathcal{E} \cap B_n \text{ & edges between any two pairs \( \delta \) points within distance } 2\mathcal{E} \).

Remark: All results stated here hold for \( \mathcal{G}_n(\mathcal{E}) \) with vertex set \( \frac{1}{\sqrt{n}} \mathcal{E} \cap B_n \text{ & } \mathcal{G} = \frac{1}{\sqrt{n}} \).

\[ N_\infty(B_n) = \# \text{ points in } \mathcal{E} \cap B_n \text{ that are part } \delta \text{ of the infinite component in the Boolean model with vertex set } \mathcal{E} \text{ & radius } \mathcal{E} \]

Call this graph \( G_n(\mathcal{E}) \)

Prop. \( \Theta(r) = 1^0 \) (origin percolates)

\[ = E[N_\infty(B_1)] \]

Pf. Apply Campbell-Mecke formula.

Def. For any \( \alpha \in (0,1) \), \( G_n(\mathcal{E}) \) is said to be \( \alpha \)-almost connected if it contains a connected component \( C \) of at least \( \alpha n \) vertices.

Thm: Let \( \alpha^* = \inf \{ r : \Theta(r) > \alpha^2, \alpha \in (0,1) \} \)

If \( \alpha > \alpha^* \) then \( G_n(\mathcal{E}) \) is \( \alpha \)-almost connected a.a.s. & if \( \alpha < \alpha^* \) it is not.

Full Connectivity

\( \mathcal{E} \subset \text{PPP}{(1)} \) on \( \mathbb{R}^2 \)

\( B_n = \left[ 0, \frac{\pi}{\sqrt{n}} \right]^2 \)

\( \mathcal{E}_n = \mathcal{E} \cap B_n \)

To avoid edge effects, we will take metric on \( B_n \) to be the toroidal metric:

\[ d_n(x,y) = \inf \{ d(x, y+z) \} \quad \forall x, y \in \mathbb{R} \]

\( G_n(\mathcal{E}) = \text{graph with vertex set } \mathcal{E} \cap B_n \text{ & edges between any two pairs of points vertices at dist. } \delta \leq 2\mathcal{E} \).

\( W_n = \# \{ \text{ isolated nodes in } G_n(\mathcal{E}) \} \)

\[ E[W_n] = n \cdot \frac{\pi}{(2\pi)^2} \]

So if \( \frac{\pi}{(2\pi)^2} = \log n + \alpha \), \( \alpha \rightarrow \alpha^* \) then \( E[W_n] \rightarrow e^{-\alpha} \)

Remark: 1. For dense network, we should take \( e^{-\alpha} = \log n + \alpha \)

2. \( E[W_n] \rightarrow e^{-\alpha} \) of chance any node being isolated is tending to 0. Dependence is local.

Thm: If \( \pi(2\pi)^2 = \log n + \alpha \), \( \alpha \rightarrow \alpha^* \)

Then \( W_n \rightarrow Po(e^{-\alpha}) \)

\[ TP(W_n = 0) \rightarrow e^{-e^{-\alpha}}, \alpha \in \mathbb{R} \]

Remark: Thus to eliminate isolated nodes we must have \( \alpha \rightarrow \infty \). Remarkably this suffices to connect the graph!!

Thm: If \( \pi(2\pi)^2 = \log n + \alpha \), then \( G_n(\mathcal{E}) \) is connected a.a.s. & if \( \alpha \rightarrow \infty \)
Interference Limited Networks

Static SINR Graph
(without fading)

\[ \bar{\Phi} \sim HPPP(\lambda) \text{ on } TR^2 \]

\[ l : TR^2 \times TR^2 \to TR_+ \text{ such that} \]

\[ l(x, y) = h(|x - y|) \text{ for some} \]

\[ h : TR_+ \to TR_+ \text{, cont. strictly decreasing on the set where it is positive} \]

\[ \int_0^\infty h(x) dx < \infty \; ; \; h(0) = 1 + \frac{1}{2} \]

\[ l : \text{path loss fn.} \]

\[ \text{Let } P, T, N \text{ be parameters} \]

\[ \gamma > 0. \]

\[ \gamma > \frac{TN}{P} \]

\[ I_{xy} = \sum_{\xi \in \Phi \setminus \{x, y\}} P l(|x - y|) \]

\[ R_{xy} = \frac{P l(|x - y|)}{N + \gamma I_{xy}} , \; xy \in \Phi. \]

Def. The static SINR graph with vertex set \( \Phi \) and directed edge set

\[ E = \{ (x, y) : x, y \in \Phi, R_{xy} > T^3 \} \]

\[ |C| = \# \text{ points } \Phi \text{ in the cluster (component) containing the origin under } P^o. \]

Graph is said to percolate if

\[ P^o(\text{containing a point}) > 0. \]

Note: i) Length of edges unit.

\[ \text{bounded by } R(\frac{TP^1}{P}) \]

\[ \gamma = 0 \text{ reduces the model to a standard Boolean model.} \]

iii) \( \gamma = 0 \) & fading we get the random connection model.

Prop. - For any \( \gamma > 0 \) any node in the static SINR graph is connected to at most \( 1 + \frac{1}{2} \) neighbours.

Thm. - Let \( \lambda_c \) be the critical node density for the graph to percolate when \( \gamma = 0 \). For any node density \( \lambda > \lambda_c \), there exists a \( \gamma^*(\lambda) > 0 \) such that for any \( \gamma < \gamma^*(\lambda) \) the static SINR graph percolates.

Thm. - For \( \lambda \to \infty \) we have that \( \gamma^*(\lambda) = O(\frac{1}{\lambda}) \).

So as the intensity increases the interference effect dominates the opposing effect that facilitates formation of edges due to large availability of nodes.