Strong Converse for a Degraded Wiretap Channel via Active Hypothesis Testing

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Abstract—We establish an upper bound on the rate of codes for a wiretap channel with public feedback for a fixed probability of error and secrecy parameter. As a corollary, we obtain a strong converse for the capacity of a degraded wiretap channel with public feedback. Our converse proof is based on a reduction of active hypothesis testing for discriminating between two channels to coding for wiretap channel with feedback.

I. INTRODUCTION

We consider secure message transmission over a wiretap channel $W : \mathcal{X} \to \mathcal{Y} \times \mathcal{Z}$ with no feedback. For each transmission $x \in \mathcal{X}$ over $W$, the receiver observes a random output $y \in \mathcal{Y}$ and an eavesdropper observes a correlated side-information $z \in \mathcal{Z}$, with probability $W(y, z|x)$. Furthermore, the receiver can send a feedback to the transmitter over a noiseless channel. However, the feedback channel is public and any communication sent over it is available to the eavesdropper. The transmitter seeks to send a message $M$ to the receiver without revealing it to the eavesdropper. For a given probability of error $\epsilon$ and a given secrecy parameter $\delta$, what is the maximum possible rate $C_{\epsilon, \delta}$ of a transmitted message?

For a degraded wiretap channel $W$ with no feedback, the wiretap capacity $C = \inf_{\epsilon, \delta} C_{\epsilon, \delta}$ was established in the seminal work of Wyner [19] where it was shown that

$$C = \max_{P_X} I(X \wedge Y | Z).$$

The capacity of a general wiretap channel was established in [3]. Extensions to wiretap channels with general statistics were considered in [4]. The model with feedback considered here was introduced in [8] where it was noted that the availability of the noiseless feedback can enable positive rates of transmission over a wiretap channel with zero capacity (see, also, [10]). However, the wiretap capacity with feedback remains unknown in general; $\max_{P_X} I(X \wedge Y | Z)$ constitutes an upper bound on it.

In this paper, we establish a strong version of this bound and show that for $\epsilon + \delta < 1$

$$C_{\epsilon, \delta} \leq \max_{P_X} I(X \wedge Y | Z),$$

thereby characterizing $C_{\epsilon, \delta}$ for all $0 < \epsilon, \delta < 1$ for a degraded wiretap channel. A partial strong converse for a degraded wiretap channel was established in [11] for a restricted range of $\epsilon, \delta$. Another strong converse for a degraded wiretap channel for the case when $\delta \to 0$ was established, concurrently to this work, in [15]. In this work, we show a strong converse for all values of $\epsilon$ and $\delta$.

Our proof relies on a slight modification of a recent reduction of hypothesis testing to secret key agreement shown in [17], [18]. Specifically, we show that a wiretap channel code yields an active hypothesis test for distinguishing between two channels [6]. Consequently, the rate of a wiretap code is bounded above by the rate of the optimum exponent of the probability of error of type II for discriminating a channel $W$ from another channel $V$ such that $V(y, z|x) = V'_2(z|x)V'_1(y|z)$, given that the probability of error of type I is less than $\epsilon + \delta$. This gives an upper bound on the length of a wiretap code, which leads to the strong converse upon using the characterization of the optimal exponent for channel discrimination derived in [6]. This approach is along the lines of meta-converse of [13], where a reduction of hypothesis testing to channel coding was used to establish a finite-blocklength converse for the channel coding problem (see, also, [12] and [5, Section 4.6]).

Our main result is given in the next section. Section III and IV contains a review of relevant results in binary hypothesis testing and secret key agreement, respectively. The final section contains a proof of our main result.

II. MAIN RESULT

We describe a generalization of the classic wiretap channel coding problem [19], [3] that was considered in [8], [10], [1], where, in addition to transmitting over the wiretap channel, the terminals can communicate using a noiseless, public feedback channel from the receiver to the transmitter.

A wiretap code for a discrete1 memoryless wiretap channel $W : \mathcal{X} \to \mathcal{Y} \times \mathcal{Z}$ with feedback consists of (possibly randomized) encoder mappings $e_t : \{1, \ldots, N\} \times \mathcal{F}^t \to \mathcal{X}$, $1 \leq t \leq n$, feedback mappings $f_t : \mathcal{Y}^t \to \mathcal{F}$, $0 \leq t \leq n - 1$, and a decoder $d : \mathcal{Y}^n \to \{1, \ldots, N\}$. For a random message $M \sim \text{unif} \{1, \ldots, N\}$, the protocol begins with a feedback $F_0$ from the receiver at $t = 0$. Subsequently, at each time instant $1 \leq t \leq n - 1$ the transmitter sends $X_t = e_t(M, F^{t-1})$ and the channel outputs $(Y_t, Z_t)$

1The restriction to discrete alphabet is cosmetic. Our results apply to channels with continuous alphabet. In particular, our strong converse holds for the Gaussian wiretap channel [9].
with probability $W(Y_t, Z_t | X_t)$. The receiver observes $Y_t$ and sends feedback $F_t = f_t(Y^t)$, and the eavesdropper observes $Z_t$. The protocol stops with a final transmission $X_n = e_n(M, F^{n-1})$ over the channel and the subsequent decoding $M = d(Y^n)$ by the receiver. We denote by $F$ the overall feedback communication $F_0, ..., F_{n-1}$.

The mappings $(\{e_t\}_{t=1}^n, \{f_t\}_{t=0}^{n-1}, d)$ constitute an $(N, n, \epsilon, \delta)$ wiretap code if

$$P(M \neq \hat{M}) \leq \epsilon,$$

and

$$\|P_{MZ^nF} - P_M \times P_{Z^nF}\|_1 \leq \delta,$$

where $\|P - Q\|_1$ denotes the variation distance between $P$ and $Q$ given by

$$\|P - Q\|_1 = \frac{1}{2} \sum_x |P(x) - Q(x)|.$$

A rate $R > 0$ is $(\epsilon, \delta)$-achievable if there exists an $(2^nR, n, \epsilon, \delta)$ wiretap code for all $n$ sufficiently large. The $(\epsilon, \delta)$-wiretap capacity $C_{\epsilon, \delta}$ is the supremum of all $(\epsilon, \delta)$-achievable rates.

Our main result in an upper bound on $C_{\epsilon, \delta}$

**Theorem 1.** For $0 \leq \epsilon, \delta$ with $\epsilon + \delta < 1$, the $(\epsilon, \delta)$-wiretap capacity is bounded above as

$$C_{\epsilon, \delta} \leq \max_{P_X} I(X \land Y | Z).$$

For the special case of a degraded wiretap channel $W$ with $W(y, z | x) = W_1(y | x)W_2(z | y)$, Theorem 1 yields a strong converse for wiretap capacity.

**Corollary 2.** For a degraded wiretap channel $W$,

$$C_{\epsilon, \delta} = \begin{cases} \max_{P_X} I(X \land Y | Z), & 0 < \epsilon < 1 - \delta, \\ \max_{P_X} I(X \land Y), & 1 - \delta \leq \epsilon < 1. \end{cases}$$

**Proof.** For $0 < \epsilon < 1 - \delta$, the result is an immediate corollary of Theorem 1 and [19]. For $1 - \delta \leq \epsilon < 1$, the converse follows from the strong converse for the capacity of a DMC with feedback (cf. [14]). Moving to the proof of achievability, it suffices to restrict to $\epsilon + \delta = 1$. For this case, achievability follows by randomizing among an $(\epsilon_n, 1)$ wiretap code, $\epsilon_n \to 0$ as $n \to \infty$, and a $(1, 0)$ wiretap code – the randomizing bit is communicated as the public feedback $F_0$ by the receiver.

As a preparation for the proof of Theorem 1 given in Section V, we review some results in hypothesis testing and secret key agreement in the next two sections.

**III. HYPOTHESIS TESTING**

Consider a simple binary hypothesis testing problem with null hypothesis $P$ and alternative hypothesis $Q$, where $P$ and $Q$ are distributions on the same alphabet $\mathcal{X}$. Upon observing a value $x \in \mathcal{X}$, the observer needs to decide if the value was generated by the distribution $P$ or the distribution $Q$.

To this end, the observer applies a stochastic test $T$, which is a conditional distribution on $\{0, 1\}$ given an observation $x \in \mathcal{X}$. When $x \in \mathcal{X}$ is observed, the test $T$ chooses the null hypothesis with probability $T(0|x)$ and the alternative hypothesis with probability $T(1|x) = 1 - T(0|x)$. For $0 \leq \epsilon < 1$, denote by $\beta_\epsilon(P, Q)$ the infimum of the probability of error of type II given that the probability of error of type I is less than $\epsilon$, i.e.,

$$\beta_\epsilon(P, Q) := \inf_{T : P[T] \geq 1 - \epsilon} Q[T],$$

where

$$P[T] = \sum_x P(x)T(0|x),$$

$$Q[T] = \sum_x Q(x)T(0|x).$$

The following result credited to Stein characterizes the optimum exponent of $\beta_\epsilon(P^n, Q^n)$ where $P^n = P \times ... \times P$ and $Q^n = Q \times ... \times Q$.

**Lemma 3.** (cf. [7, Theorem 3.3]) For every $0 < \epsilon < 1$, we have

$$\lim_{n \to \infty} -\frac{1}{n} \log \beta_\epsilon(P^n, Q^n) = D(P || Q),$$

where $D(P || Q)$ is the Kullback-Leibler divergence given by

$$D(P || Q) = \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)},$$

with the convention $0 \log(0/0) = 0$.

Next, we review a problem of active hypothesis testing where the distribution at each instance is determined by a prior action. Specifically, given two DMCs $W : \mathcal{X} \to \mathcal{Y}$ and $V : \mathcal{X} \to \mathcal{Y}$, we seek to design a transmission-feedback scheme such that by observing the channel inputs, channel outputs, and feedback we can determine if the underlying channel is $W$ or $V$. Formally, an $n$-length active hypothesis test consist of (possibly randomized) encoder mappings $e_t : \mathcal{F}^t \to \mathcal{X}$, $1 \leq t \leq n$, feedback mappings $f_t : \mathcal{Y}^t \to \mathcal{F}$, $0 \leq t \leq n - 1$, and a conditional distribution $T$ on $\{0, 1\}$ given $X^n, Y^n, F$. On observing $X^n, Y^n, F$, we detect the null hypothesis $W$ with probability $T(0|X^n, Y^n, F)$ and alternative hypothesis $V$ with probability $T(1|X^n, Y^n, F)$. Analogous to $\beta_\epsilon(P, Q)$, the quantity $\beta_\epsilon(W, V, n)$, for $0 \leq \epsilon < 1$, is the infimum of the probability of error of type II over all $n$ length active hypothesis tests for null hypothesis $W$ and alternative hypothesis $V$ such that the probability of error of type I is no more than $\epsilon$.

The following analogue of Stein’s lemma for active hypothesis testing was established in [6] (see, also, [14]).
Theorem 4 ([6]). For $0 < \epsilon < 1$, 
\[
\lim_{n} \frac{1}{n} \log \beta_{\epsilon}(W, V, n) = \max_{P_X} D(W \| V | P_X) = \max_{z} D(W_z \| V_z),
\]
where $W_x$ and $V_x$, respectively, denote the $x$th row of $W$ and $V$.

Remarkably, the exponent above is achieved without any feedback, i.e., while feedback is available, it does not help to improve the asymptotic exponent of $\beta_{\epsilon}(W, V, n)$.

IV. SECRET KEY AGREEMENT

In this section, we review two party secret key (SK) agreement where parties observing random variables $X$ and $Y$ communicate interactively over a public channel to agree on a SK that is concealed from an eavesdropper with access to the communication and a side-information $Z$.

Formally, the parties communicate using an interactive communication $\mathbf{F} = F_1, \ldots, F_r$ where $F_1 = F_1(X), F_2 = F_2(Y, F_1), F_3 = F_3(X, F_2), F_4 = F_4(Y, F_3)$ and so on. A random variable $\hat{K} = \hat{K}(X, \mathbf{F})$ constitutes an $(\epsilon, \delta)$-SK if there exists $\hat{K} = \hat{K}(Y, \mathbf{F})$ such that
\[
P(\hat{K} \neq \hat{K}) \leq \epsilon,
\]
and
\[
\lVert P_{KZ} - P_{\text{unif}} \times P_{ZF} \rVert_1 \leq \delta.
\]
The following upper bound on the number of values $k$ taken by an $(\epsilon, \delta)$-SK $K$ was shown in [17], [18]:
\[
\log k \leq -\log \beta_{\epsilon+\delta+\eta}(P_{XYZ}, Q_{XYZ}) + 2 \log \frac{1}{\eta},
\]
for all $0 < \eta < 1 - \epsilon - \delta$, and all $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_{Z}$. Underlying the proof of this bound is an intermediate reduction argument in [17, Lemma 1] that relates SK agreement to hypothesis testing. We recall this result below.

Theorem 5 ([17], [18]). For $0 \leq \epsilon, \delta, \epsilon + \delta < 1$, let random variables $K, \hat{K}$, and $Z$ be such that $P(\hat{K} \neq \hat{K}) \leq \epsilon$ and
\[
\lVert P_{KZ} - P_{\text{unif}} \times P_{Z} \rVert_1 \leq \delta,
\]
where $P_{\text{unif}}$ denotes a uniform distribution on $k$ values. Then, for every $0 < \eta < 1 - \epsilon - \delta$ and every $Q_{K|KZ} = Q_{K|Z}Q_{K|Z}Q_{Z}$,
\[
\log k \leq -\log \beta_{\epsilon+\delta+\eta}(P_{KKZ}, Q_{KKZ}) + 2 \log \frac{1}{\eta}.
\]

V. PROOF OF MAIN RESULT

We present a converse result that applies for every fixed $n$ and is asymptotically tight, giving the strong converse result of Theorem 1.

Theorem 6. For $0 \leq \epsilon, \delta, \epsilon + \delta < 1$, given an $(N, n, \epsilon, \delta)$-wiretap code, we have
\[
\log N \leq -\log \beta_{\epsilon+\delta+\eta}(W, V, n) + 2 \log \frac{1}{\eta},
\]
for all $0 < \eta < 1 - \epsilon - \delta$ and all channels $V : X \rightarrow Y \times Z$ such that $V(y, z|x) = V_2(z|x)V_1(y|z)$.

Proof of Theorem 1. Theorem 1 follows form Theorems 6 and 4 upon noting that for $W(y, z|x) = W_2(z|x)W_1(y, x)$
\[
\min \max_{V} D(W \| V | P_X) = \min \max_{V, V_1} D(W_1 \| V_1 | P_XW_2) = \max_{V} \min \max_{V_1} D(W_1 \| V_1 | P_XW_2) = \max_{V} D(P_{Y|Z}|P_{Y}| P_{ZX}) = \max_{V} I(X \wedge Y | Z),
\]
where $P_{XYZ}$ is given by $P_XW$.

We need the following result to prove Theorem 6.

Lemma 7. For a wiretap channel $V : X \rightarrow Y \times Z$ such that $V(y, z|x) = V_2(z|x)\nu_1(y|z)$, a random message $M$, and a wiretap code, let $\hat{M} = d(Y^n)$ and $\mathbf{F}$ be the corresponding feedback. Then, the induced distribution $Q_{\hat{M}|Z^n|F^n}$ satisfies factorization condition
\[
Q_{\hat{M}|Z^n|F^n} = Q_{M|Z^n|F^n} \times Q_{\hat{M}|Z^n|F^n}.
\]

Proof of Lemma 7. Denote by $U_x$ and $U_y$, respectively, the local randomness at the transmitter and the receiver, and by $F_t$ the feedback $(F_0, \ldots, F_t)$. Thus, the encoder mapping $e_t$ is a (deterministic) function of $(M, U_x, F_t)$ and the feedback mapping $f_t$ is a (deterministic) function of $(Y^t, U_y)$. The proof entails a repeated application of the fact that conditionally independent random variables remain so when conditioned additionally on an interactive communication ([16]) and is completed by induction. Specifically, note first that $Q_{\hat{M}|U_x, U_y, \nu_t} = Q_{\hat{M}|U_x, \nu_t}Q_{U_y|U_x, \nu_t}$ since $(M, U_x)$ and $U_y$ are independent and $F_0$ is an interactive communication. Under the induction hypothesis
\[
Q_{\hat{M}|U_x, X_t-1, U_y, Y_t-1|Z_t-1, F_t-1} = Q_{\hat{M}|U_x, X_t-1, Y_t-1|Z_t-1, F_t-1}Q_{U_y|X_t-1, Y_t-1|Z_t-1, F_t-1},
\]
we get
\[
I(M, U_x, X_t \wedge U_y, Y_t \mid Z_t, F_t-1) = I(M, U_x, X_t \wedge U_y, Y_t-1 \mid Z_t, F_t-1) \leq I(M, U_x, X_t \wedge U_y, Y_t-1 \mid Z_{t-1}, F_t-1) = I(M, U_x, X_t-1 \wedge U_y, Y_t-1 \mid Z_{t-1}, F_t-1) = 0,
\]
where the first equality and inequality follow since $Y_t$ and $Z_t$, respectively, are outputs of $V_1$ for input $U_x$ and $V_2$ for input $X_t$, and the second equality holds since $X_t = e_t(M, U_x, F_t-1)$, which completes the proof.

Proof of Theorem 6. Given an $(N, n, \epsilon, \delta)$-wiretap code, a message $M \sim \text{unif}\{1, \ldots, N\}$ and its decoded value $\hat{M} = d(Y^n)$ satisfy the conditions for Theorem 5 with $K = M, \hat{K} = \hat{M}$, and $Z = (Z^n, \mathbf{F})$. Letting $Q_{\hat{M}|Z^n|F^n}$ be the distribution on $(M, \hat{M}, Z^n, \mathbf{F})$ when the underlying
channel is $V$, by Lemma 7 and Theorem 5 we get
\[ \log N \leq - \log \beta_{e+\delta+n}(P_{\hat{M}Z^nF}, Q_{\hat{M}Z^nF}) + 2 \log \frac{1}{\eta}. \]

Note that a test for the simple binary hypothesis testing problem for $P_{\hat{M}Z^nF}$ and $Q_{\hat{M}Z^nF}$ along with the wiretap code constitutes an active hypothesis test for $W$ and $V$. Therefore,
\[ - \log \beta_{e+\delta+n}(P_{\hat{M}Z^nF}, Q_{\hat{M}Z^nF}) \leq - \log \beta_{e+\delta+n}(W, V, n), \]

which completes the proof. \hfill \Box

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