Lecture 8 (Cont'd) 

Proof Sketch of the claim ①

\[ \mathbb{E}_i [\mathbb{E}_i (Z)] = \mathbb{E} \left[ \mathbb{E} [Z | X^{-i}] \right] | X^{i} \] 

function only of \( X^{-i} \)

Let \( Z = f(A, B, C) \), \( A, B, C \) indep.

\[ \mathbb{E} \left[ \mathbb{E}[Z | AC] | AB \right] = \mathbb{E} \left[ \mathbb{E}[Z | A] | AB \right] = \mathbb{E}[Z | A] \]

Other equivalent forms of Efron-Stein

Let \( \nu = \sum_{i=1}^{n} \mathbb{E} \left[ \text{Var}_i (Z) \right] \).

Efron-Stein implies that \( \nu \) can be treated as the effective variance of \( Z \). The good thing is that \( \nu \) can be expressed as the sum of individual contributions across \( i = 1, 2, \ldots, n \). The next result gives two useful alternative forms of \( \nu \).

Lemma. (a) Let \( X' = (x_1', \ldots, x_n') \) be an independent copy of \( X \). Then,

\[ \nu = \sum_{i=1}^{n} \mathbb{E} \left[ (Z - Z_i')^2 \right] \]

where \( Z_i' = f(X_i, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n) \).

(b) \( \nu = \inf_{\{Z_i\}} \mathbb{E} \left[ (Z - Z_i)^2 \right] \),

where the inf. is over all square integrable func. of
Proof: Follows from the following 2 simple facts about variances:

(i) $X$ and $Y$ be iid

$$\text{Var}(X) = \frac{1}{2} \left[ \text{IE} [(X-Y)^2] \right]$$

(ii) $\text{Var}(X|Y) = \text{IE} \left[ (X - \text{IE}(X|Y))^2 \right]$

$$= \inf_{Z} \frac{\text{IE} \left[ (X - Z)^2 \right]}{2} = \text{MMSE}(X|Z)$$

where the infimum is over all functions $Z$ of $Y$.

Applying these observations to $\text{Var}(x_i, (Z))$ gives the result.

Corollary: If $f$ satisfies BDP with $(c_1, ..., c_n)$

Then, $\text{Var}(f(x_1, ..., x_n)) \leq \sum_{i=1}^{n} \frac{c_i^2}{4}$

Proof: Use (b) with

$Z; i = \frac{1}{2} \left[ \sup_{x_i \in X_i} f(x_i, x, x_i) + \inf_{x_i \in X_i} f(x_i, x, x_i) \right]$

Example: $Z = \text{length of the longest incr. subseq.}$

of $x_1, ..., x_n$

$\text{Var}(Z) \leq n \Rightarrow \text{P} \left( |Z - \text{IE}(Z)| > \sqrt{\frac{n}{\delta}} \right) \leq \delta$. 
The first and the second moment methods

\[ P(X \neq 0) = P(X \geq 1) \leq \mathbb{E}[X] \]

The "first-moment method"

**Example:** Consider an n-uniform hypergraph with m edges. If \( m < 2^{n-1} \) then \( \mathcal{H} \) is 2-colorable, i.e., there exists a coloring of vertices using 2 colors s.t. no hyperedge is monochromatic.

Indeed, consider a random 2-coloring of \( \mathcal{H} \) where each vertex is independently and uniformly colored using red/blue color. Let \( X \) denote the # of monochromatic edges. Then,

\[ P(X \neq 0) \leq \mathbb{E}[X] = \sum_{e \in \mathcal{E}} P(e \text{ is monochromatic}) \]

\[ = m \cdot 2^{-n+1} < 1. \]

For a r.v. \( X \) with \( \mathbb{E}[X] \neq 0, \)

\[ P(X \leq 0) \leq P(|X - \mathbb{E}[X]| > \mathbb{E}[X]) \leq \frac{\text{Var}(X)}{[\mathbb{E}[X]]^2} \]

**Improvement:** \( \mathbb{E}[X]^2 = \mathbb{E}[X \mathbb{1}_{X \neq 0}]^2 \)
\[ \leq \mathbb{E}[X^2](1 - \mathbb{P}(X = 0)) \]

\[ \Leftrightarrow \mathbb{P}(X = 0) \leq \frac{\text{Var}(X)}{\mathbb{E}[X]^2} \quad (\text{Shepp's bound}) \]

The "second moment method"

The bounds above give bounds for the tail \( \mathbb{P}(X > 0) \) for nonnegative rv's.

Concentration around median using Efron-Stein

Quantiles of a distribution:

\[ Q_\alpha = \inf \{ z : \mathbb{P}(Z \leq z) \geq \alpha \} : \alpha \text{-th quantile of } Z \]

\[ M_Z \equiv \text{median of } Z = Q_{\frac{1}{2}} \]

Let \( q_k \) denote \( Q_{1-2^{-k}} \), i.e., \( \mathbb{P}(Z > q_k) \leq 2^{-k} \).

(i) \( \lim_{k \to \infty} q_k = \sup Z \)

(ii) Suppose \( q_{k+1} - q_k \leq c \) for every \( k \in \mathbb{N} \).

Then,

\[ \mathbb{P}(Z > M[Z] + t) \leq 2^{-\frac{t}{c}} \]

Proof: Let \( k_t \) denote the \( k \) s.t.

\[ q_{k_t} \leq t + q_1 \leq q_{k_t+1} \]

Then, \( q_{k_t+1} = q_1 + \sum_{i=1}^{k_t} (q_{i+1} - q_i) \leq q_1 + c k_t \)

\[ \Rightarrow t \leq c k_t \]
Thus,

\[ P(Z > M[z] + t) \leq P(Z > q_{k_t}) \]
\[ \leq 2^{-k_t} \leq 2^{-t/c}. \]