Self-Bounding Function

Definition. A function \( f: \mathbb{R}^n \rightarrow [0, \infty) \) has the self-bounding property if there exist \( f_i(x_1, \ldots, x_i, \ldots, x_n) \) s.t.

1. \( 0 \leq f(x) - f_i(x^i) \leq 1 \),
2. \( \sum_{i=1}^{n} (f(x) - f_i(x^i)) \leq f(x) \).

Clearly, for self-bounding functions

\[
\sum_{i=1}^{n} (f(x) - f_i(x^i))^2 \leq f(x)
\]

Corollary. For a self-bounding function \( f \), \( Z = f(x) \) satisfies

\[
\text{Var}(Z) \leq \mathbb{E}[Z^2].
\]

Definition (hereditary property, configuration functions)

A property \( \Pi \) is said to be hereditary if when a sequence satisfies it, any subsequence satisfies it as well.

A configuration function is given by \( f(x_1, \ldots, x_n) = \text{length of the longest subsequence satisfying a hereditary property } \Pi \).

Lemma. A configuration function is self-bounding.

Proof. Let \( f_i(x^i) = f(x_1, \ldots, x_i, \ldots, x_n) \). Then,

\[
0 \leq f(x) - f_i(x^i) \leq 1.
\]
Note that for a seq. $x = (x_1, \ldots, x_n)$ where the longest subseq. satisfying $P_i$ is $(x_i, \ldots, x_{i+k})$, for every $i \neq \epsilon_1, \ldots, \epsilon_k$,
\[ f(x) - f_i(x) = 0. \]
Therefore,
\[ \sum_{i=1}^{n} f(x) - f_i(x) \leq k = f(x). \]

**Examples of self-bounding functions**

* longest increasing subsequence
* no. of distinct elements

**Example (Maximum eigenvalue)**

Consider a random matrix $A$ with entries $X_{ij}$, $1 \leq i \leq j \leq n$, that are iid with $|X_{ij}| \leq 1$. Let $Z$ denote the max. eigenvalue of $A$, i.e.,
\[ Z = \sup_{u: \|u\|=1} u^T A u. \]
Then, \( (Z - z_{ij})_+ \leq (v^T A u - v^T A_{ij} v) \mathbb{1}_{Z > z_{ij}} \)
\[ = (v^T (A - A_{ij}) v) \mathbb{1}_{Z > z_{ij}} \]
\[ \leq 2 \langle v, v \rangle (x_{ij} - x_{ij})_+ \]
\[ \leq 4 |v, v|, \]
where $v$ achieves $Z$ for $A$. Thus,
\[ \sum_{1 \leq i < j \leq n} (Z - z_{ij})_+ \leq 16 \sum_{1 \leq i < j \leq n} |v, v|^2 \leq 16. \]
The Entropy Method

Recall the recipe we used to show the concentration of $Z$ around its mean using the Efron-Stein inequality:

\[ \text{Var} \left( e^{\frac{Z^2}{2}} \right) \leq \frac{\lambda}{q} \mathbb{E} \left[ e^{\frac{Z^2}{2}} \sum_{i=1}^{n} (Z - Z_i)^2 \right] \]

which implied

\[ \psi_{Z-\mathbb{E}Z}(\frac{1}{\sqrt{q}}) \leq \log \frac{16}{\frac{\lambda}{q}} \]

Entropy method generalizes this recipe (we shall see in what sense) and yields stronger results than what we obtained earlier.

[ ] Step 1: Entropy and Herbst's argument

\[ \text{Var}(X) = \mathbb{E} \left[ g(X) \right] - g(\mathbb{E}[X]) \]

for $g(x) = x^2$.

Consider an alternative concave function $h(x) = x \log x$. Let $x > 0$ a.s.

\[ \text{Ent}(x) \overset{\text{def}}{=} \mathbb{E} \left[ h(X) \right] - h(\mathbb{E}[X]) \]
Lemma (Herbst's argument)

Let \( Z \) be a rv with \( \mathbb{E}[Z] < \infty \) s.t. there exists \( \nu > 0 \) for which

\[
(\#) \quad \frac{\mathbb{E}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda^2 Z/2}]} \leq \frac{\lambda^{2 \nu}}{2}, \quad \forall \lambda > 0.
\]

Then, \( \forall \lambda > 0 \)

\[
\psi_{\mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^{2 \nu}}{2}.
\]

Proof. The key observation is the following:

\[
\psi'(\lambda) = \frac{\mathbb{E}[Ze^{\lambda Z}]}{\mathbb{E}[e^{\lambda^2 Z/2}]},
\]

which implies

\[
\frac{\mathbb{E}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda^2 Z/2}]} = \lambda \psi'(\lambda) - \psi(\lambda).
\]

Therefore, the Herbst's condition \((\#)\) is the same as

\[
\lambda \psi'(\lambda) - \psi(\lambda) \leq \frac{\lambda^{2 \nu}}{2},
\]

i.e., for \( G(\lambda) = \frac{\psi(\lambda)}{\lambda} \),

\[
G'(\lambda) \leq \frac{\nu}{2} \Rightarrow G(\lambda) - G(0) \leq \frac{\lambda^{2 \nu}}{2}.
\]

\[
G(0) = \lim_{\lambda \to 0} \frac{\psi(\lambda)}{\lambda} = \frac{\psi'(0)}{0} = 0.
\]

Thus, \( \psi(\lambda) \leq \frac{\lambda^{2 \nu}}{2} \).

\[\square\]
Step 2: Tensorization of HerbSTE argument

Suppose that we can establish (♯) for \( n = 1 \). Then,
\[
\frac{\text{Ent}^{(i)}(e^{\lambda x})}{\mathbb{E}^{(i)}[e^{\lambda x}]} \leq \frac{\lambda^2 \nu_i}{2} \left( \text{Ent}^{(i)}(y) \right) \frac{\lambda^2 \nu_i}{2} \left( \mathbb{E}^{(i)}[\mathbb{E}^{(i)}[y]] - h(\mathbb{E}^{(i)}[y]) \right)
\]

Suppose the counterpart of ES for \( \text{Var}(\cdot) \) holds, i.e.,
\[
\text{Ent}(e^{\lambda x}) \leq \sum_{i=1}^{n} \mathbb{E} \left[ \text{Ent}^{(i)}(e^{\lambda x}) \right].
\]

Then, the bound for \( n = 1 \) yields
\[
\text{Ent}(e^{\lambda x}) \leq \sum_{i=1}^{n} \mathbb{E} \left[ \frac{\lambda^2 \nu_i}{2} \mathbb{E}^{(i)}[e^{\lambda x}] \right]
\]
\[
= \frac{\lambda^2 \nu_i}{2} \left( \sum_{i=1}^{n} \right).
\]