(1) **Distinct values in an iid sequence**

Let $X_1, \ldots, X_n$ be iid random variables taking positive integer values, and let $Z_n$ be the number of distinct values taken by these random variables.

(a) Show that $\lim_{n \to \infty} \mathbb{E} \left[ \frac{Z_n}{n} \right] = 0$, or in other words, $\mathbb{E} \left[ \frac{Z_n}{n} \right] = o(n)$.

(b) Does $Z_n$ satisfy the bounded differences property? If so, what can you conclude about the variance of $Z_n$ (as a function of $n$)?

(c) Show that $Z_n$ is in fact a self-bounding function. What can you now conclude about its variance as a function of $n$? Compare this to the conclusion of the previous part.

(2) **Order Statistics and variance**

Let $X_1, \ldots, X_n$ be a sequence of independent random variables, and $X_{(1)} \leq X_{(1)} \leq \cdots \leq X_{(n)}$ a sorted version of the sequence ($X_{(i)}$ is known as the $i$-th order statistic).

(a) Show that $\text{Var}[X_{(n)}] \leq \mathbb{E} \left[ \left( X_{(n)} - X_{(n-1)} \right)^2 \right]$.

(b) Compute the LHS and RHS of the inequality above, when all the $X_i$ are iid Exponential(1).

(c) Repeat for all the $X_i$ iid Uniform([0, 1]).

(3) **Jackknife estimators**

Let $X_1, \ldots, X_n$ be an iid sequence of random variables (a “sample” of size $n$ in statistics terminology). Suppose one has designed, for any $n$, an estimator $T_n \equiv T_n(X_1, \ldots, X_n)$ for a scalar parameter $\theta \in \mathbb{R}$ of the common probability distribution of the $X_i$. One often wants to know the quality of the estimator $T_n$ (using only the sample). The jackknife is a method to estimate the bias $\mathbb{E} \left[ T_n \right] - \theta$ and variance $\mathbb{E} \left[ \left( T_n - \mathbb{E} \left[ T_n \right] \right)^2 \right]$ of the estimator $T_n$.

For each $i \in [n]$, define the $i$th pseudo-value

$$Y_i := n T_n(X_1, \ldots, X_n) - (n - 1) T_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n).$$

The **jackknife estimate of the bias** of $T_n$ is defined to be the difference between $T_n$ and the sample mean of the pseudo-values, i.e., $\hat{B}_n := T_n - \frac{1}{n} \sum_{i=1}^n Y_i$, while the **jackknife estimate of the variance** of $T_n$ is defined to be the sample variance of the pseudo-values $Y_i$, i.e., $\hat{V}_n := \frac{1}{n-1} \sum_{i=1}^n \left( Y_i - \frac{1}{n} \sum_{j=1}^n Y_j \right)^2$. (The general principle is to imagine the pseudo-values $Y_i$ as representing “iid copies of $T_n$” and compute standard statistics on them.)

(a) Compute the jackknife estimate of the bias of the sample mean estimator $T_n := \frac{1}{n} \sum_{i=1}^n X_i$.

(b) For any estimator $T_n$, show that $\hat{V}_n$, the jackknife estimator of the variance of $T_n$, is always positively biased, i.e., $\mathbb{E} \left[ \hat{V}_n \right] - \text{Var}[T_n] \geq 0$. 


(4) Gradients of Lipschitz functions
Show that if a function \( f : \mathbb{R}^n \to \mathbb{R} \) is Lipschitz, with respect to the standard Euclidean norm on \( \mathbb{R}^n \), and differentiable, then the norm of its gradient is bounded by 1.

(5) Log-Sobolev is stronger than Poincaré
Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a bounded, continuously differentiable function. Show that the Gaussian logarithmic Sobolev inequality for \( f(X) \), with \( X \sim \mathcal{N}(0, I_{n \times n}) \), implies the Gaussian Poincaré inequality for \( f(X) \).

(6) Log-Sobolev for the exponential distribution
Let \( X \) be an exponentially distributed random variable with parameter 1, and let \( f : [0, \infty) \to \mathbb{R} \) be a continuously differentiable function. Show that \( \text{Ent}(f(X)^2) \leq 4 \mathbb{E}[X(f'(X))^2] \).