Homework Questions Q4 and Q7 are slightly difficult.

Q1 Suppose pairs of random variables $(X_1,Y_1)$ and $(X_2,Y_2)$ are independent. Show that $(X_1,Y_1)$ and $(X_2,Y_2)$ remain independent conditioned on $(f(X_1,X_2),g(Y_1,Y_2),X_1,Y_2)$ for any functions $f$ and $g$.

Q2 Show that $R_{1/3}(EQ_n) = \Theta(\log n)$.

Q3 Suppose that $P_{X,Y}(f^{-1}(z)) \geq 1/2$ for a $z \in Z$ and there exists $\alpha > 0, \delta \in (0,1)$ such that

$$\delta \geq \max \left\{ P_{X,Y} \left( R \cap f^{-1}(z) \right) : \alpha P_{X,Y} \left( R \cap f^{-1}(z) \right) > P_{X,Y} \left( R \setminus f^{-1}(z) \right), \forall R \in \mathcal{R}(X \times Y) \right\}.$$

Show that

$$D_\epsilon(f|P_{X,Y}) \geq \log \frac{1}{\delta} - \log \left( \frac{\alpha}{\alpha(0.5 - \epsilon) - \epsilon} \right).$$

Q4 Show that $D_\epsilon(DISJ_n|P_XP_Y) = O(\sqrt{n})$, for every independent distribution $P_XP_Y$ on the inputs.

Q5 Let $D_{JS}(P, Q)$ denote the Jensen-Shannon divergence between $P$ and $Q$, given by

$$D_{JS}(P, Q) = \frac{1}{2} \left( D(P\|Q) + D(Q\|P) \right).$$

Further, let $h^2(P, Q)$ denote the squared Hellinger distance between $P$ and $Q$, given by

$$h^2(P, Q) = \frac{1}{2} \sum_x (\sqrt{P(x)} - \sqrt{Q(x)})^2.$$

Show that

(i) $D_{JS}(P, Q) \geq h^2(P, Q)$.

(ii) $h^2(P, Q) \geq 1 - (1 - d^2_{TV}(P, Q))^\frac{1}{2}$.

Q6 Complete the proof of BBCR direct sum theorem; in particular, show that the expected number of disagreements in the paths simulated by the parties is no more than $\sqrt{\frac{1}{2} |\pi| IC(\pi|P_{XY})}$.

Q7 (Braverman ’12). Given a private coin protocol $\pi$ with inputs from $X \times Y$, show that when the inputs are generated by $P_{X,Y}$, $\pi$ can be $\epsilon$-simulated using no more than $2^{\tilde{O}(IC(\pi|P_{XY}))/\epsilon}$ bits of communication.