Note that in the definition of tree protocols used in the class, the leaves of the protocol are labeled with the function value (to be declared as the output).

1. [KN Exercise 1.7] Consider two parties observing subsets $x$ and $y$ of $\{1, ..., n\}$. They wish to compute the median $\text{Med}(x, y)$ of the multiset $x \cup y$. Show $L(\text{Med}) = \Theta(\log n)$.

2. [KN Exercise 1.22] Consider two parties observing $n$-bit binary strings $x$ and $y$. They wish to compute the “greater than” function $\text{GT}(x, y) = 1(x_i \geq y_i \forall i \in [n])$ of the multiset $x \cup y$. Show that $L(\text{GT}) = n + 1$.

3. [KN Exercise 1.30] Show the lower bounds of $n+1$ for deterministic communication complexity of EQ and DISJ using the rank lower bound.

4. Denote by $\text{DISJ}_{n,k}$ the set disjointness function when the input subsets of two parties are size $k$ subsets of $\{1, ..., n\}$, show that $L(\text{DISJ}_{n,k}) \geq \log \lceil n/k \rceil$.

5. [KN Exercise 3.4] Denoting by $\overline{L}(f)$ the minimum average length of a protocol that computes $f$ without error, show that $L_\epsilon(f) \leq O(\log(1/\epsilon)\overline{L}(f))$.

6. [KN Exercise 3.18] Show that $L_{1/3}^{\text{prv}}(\text{GT}) = O(\log^2 n)$ but $L_{1/3}(\text{GT}) = O(\log n)$.

7. Show Yao’s minimax theorem for communication complexity (we gave a proof sketch in the first class).

8. Show that a randomly chosen Boolean function $f : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ has $L_\epsilon(f) = \Omega(n)$ with probability close to 1.

   *Hint: This is a difficult problem. Use Yao’s minimax theorem and show that the $\Omega(n)$ lower bound holds for a random Boolean function when the inputs are uniformly chosen independent random strings. In fact, you can show that the probability that this does not hold is super-exponentially small.*

9. Recall the notation $L_{\epsilon}^{1,1}(f)$ for the randomized communication complexity of computing $f$ using a single round of communication from Party 1. Show that $L_{1/3}^{1,1}(f) = \Omega(L_{1/3}(f))$.

10. Given a function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ and a pmf $\mu$ on $\mathcal{X} \times \mathcal{Y}$, suppose that there exists another pmf $\rho$ such that for every $z \in \mathcal{Z}$ and rectangle $R \subset \mathcal{X} \times \mathcal{Y}$ it holds that

    $$\mu(R \cap f^{-1}(z)) \geq \alpha \rho(R).$$

    Then, show that

    $$L_\epsilon(f|\mu) \geq \log \frac{\alpha - \epsilon}{\alpha \delta}.$$
11. Can you use the bound of the previous exercise to establish the converse for the Slepian-Wolf result. If yes, show how. If no, modify the bound to obtain this result.

The Slepian-Wolf result considers a simple function computation problem where, for iid random variables \((X_t,Y_t)_{t=1}^n\), the first party observes \(X^n\) and the second party observes \(Y^n\), and they wish to compute \(X^n\). Slepian and Wolf showed that this can be done using \(nH(X|Y) + o(n)\) bits of communication from \(X^n\) to \(Y^n\). Furthermore, any interactive communication protocol (can we use our definition of the protocol here?) must communicate \(nH(X|Y) + o(n)\) bits.