1 Communication System

1.1 Best possible rates and probability of error

Example:

Channel $W : \mathcal{X} = \{0, 1\} \rightarrow \mathcal{Y} = \{0, 1\}$.
$M \sim Ber(0.5)$.
$W(0|0) = 0.75 = W(1|1)$.

Choose $X_1 = X_2 = M$.
Transmit $X_1, X_2$ over 2 (independent) channel uses. Receive $Y_1, Y_2$.

- $Y_1 = 0, Y_2 = 0 \quad \Rightarrow \hat{M} = 0$.
- $Y_1 = 1, Y_2 = 1 \quad \Rightarrow \hat{M} = 1$.
- $Y_1 = 0, Y_2 = 1 \quad \Rightarrow M = 0$ (arbitrary).
- $Y_1 = 1, Y_2 = 0 \quad \Rightarrow \hat{M} = 0$ (arbitrary).
\( P_e^{(2)} = Pr\{0 \text{ sent, 1 received}\} + Pr\{1 \text{ sent, 0 received}\} \)
\[= \frac{1}{2} \left[ Pr\{\hat{M} = 1|M = 0\} + Pr\{\hat{M} = 0|M = 1\} \right] \]
\[= \frac{1}{2} \left[ \frac{1}{4} \cdot \frac{1}{4} + \left( \frac{1}{4} + \frac{1}{4} - \frac{1}{16} \right) \right] \]
\[= \frac{1}{4} \]

Now let us modify the decoder: If \( Y_1 \neq Y_2 \), \( \hat{M} \sim Ber(0.5) \).
\[ P_e^{(2)} = \left[ \frac{1}{16} + \left( \frac{1}{4} + \frac{1}{4} - \frac{1}{16} \right) \frac{1}{2} \right] \]
\[= \frac{9}{32} \]

Another decoder: Majority scheme where \( \hat{M} = \text{majority}(Y_1, Y_2, ..., Y_n) \).
\[ P_e^{(n)} = \frac{1}{2} \left[ Pr\{\hat{M} = 1|M = 0\} + Pr\{\hat{M} = 0|M = 1\} \right] \]
\[= Pr\{\hat{M} = 1|M = 0\} \quad \text{(By Symmetry)} \]
\[\leq Pr\{\frac{1}{2} \text{ or more bits flipped}\} \]
\[= \sum_{i=n/2}^{n} \binom{n}{i} \left( \frac{1}{4} \right)^i \left( \frac{3}{4} \right)^{n-i} \]
\[\approx 2^{-nc}. \]

Different ways to get a bound on the probability of error:
\[ P_e^{(n)} = Pr\{N \geq n/2\} \]
\[\leq \frac{E[N]}{n/2} \quad \text{(Markov inequality)} \]
\[= \frac{1}{2}. \]

Or,
\[ P_e^{(n)} = Pr\{N \geq n/2\} \]
\[= Pr\{N - n/4 \geq n/2 - n/4\} \]
\[= Pr\{(N - n/4)^2 \geq (n/2 - n/4)^2\} \]
\[\leq \frac{Var(N)}{(n/4)^2}. \quad \text{(Chebyshev inequality)} \]

Or,
\[ P_e^{(n)} = Pr\{N \geq n/2\} \]
\[\approx 2^{-nc}. \quad \text{(Chernoff bound)} \]
Suppose we require $P_e^{(n)} \leq 0.01 = \epsilon$.

Then, we want $2^{-nc} \leq \epsilon$.

$\Rightarrow n \geq \frac{1}{c}\log \frac{1}{\epsilon}$.

Thus, we can send $\frac{c}{\log \frac{1}{\epsilon}}$ bits per channel use.

This rate goes to 0 as $\epsilon$ goes to 0. (which is not good!)

**Goal:** Find the highest number of bits per channel use possible (such that the error could be driven to zero).

### 1.2 Definitions

- **We restrict to Discrete memoryless channel.** i.e., a channel with input $\mathcal{X}$ and output $\mathcal{Y}$ discrete (in fact, $|\mathcal{X}| < \infty$, $|\mathcal{Y}| < \infty$) and given the inputs $X_1, ..., X_n$, the outputs $Y_1, ..., Y_n$ are independent.

- **Channel Code** - An $n$-length block channel code of size $2^k$ consists of an encoder $f : \{0,1\}^k \rightarrow X^n$ and a decoder $\phi : Y^n \rightarrow \{0,1\}^k \cup \{\xi\}$.

- The **rate** of this code is given by $\frac{k}{n}$ bits per channel use.

- **Maximum probability of error** for the code $(f, \phi)$,

  $$
  e_{max}(f, \phi) = \max_{m \in \{0,1\}^k} Pr\{\phi(Y_1, ..., Y_n) \neq m\} \quad \text{[where } Y_1, ..., Y_n \text{ are (random) channel outputs when } (X_1, ..., X_n) = f(m)\]
  $$

  $$
  = \max_{m \in \{0,1\}^k} \sum_{y : \phi(y) \neq m} W^n(y|f(m)).
  $$

- **Notation:** $W^n(y|x) = \Pi_{i=1}^{n} W(y_i|x_i)$.

- **Average probability of error** for $(f, \phi)$,

  $$
  e_{avg}(f, \phi) = \frac{1}{2^k} \sum_{m \in \{0,1\}^k} Pr\{\phi(Y^n) \neq m\}.
  $$