A1 Since $X$ is uniform, $H(X) = 1$. For the random variable $Y$,

\[ P(Y = 0) = \frac{1}{2} (P(Y = 0|X = 0) + P(Y = 0|X = 1)) \]
\[ = \frac{1}{2} (P(Z = 0|X = 0) + P(Z = 1|X = 1)) \]
\[ = \frac{1}{2} (P(Z = 0) + P(Z = 1)) \]
\[ = \frac{1}{2} . \]

Thus, $H(Y) = 1$. Furthermore, since $Z$ is a function of $(X, Y)$ and $Y$ is function of $(X, Z)$,

\[ H(X, Y) = H(X, Y, Z) = H(X, Z) = H(X) + H(Z) = 2, \]

where the second-last equality holds since $X$ and $Z$ are independent. It follows that $H(X) + H(Y) = H(X, Y)$ which implies that $X$ and $Y$ are independent.

Finally, since $Y$ is function of $(X, Z)$,

\[ H(X, Y|Z = 0) = H(X|Z = 0) = H(X) = 1, \]

and

\[ H(X, Y|Z = 1) = H(X|Z = 1) = H(X) = 1. \]

Thus, $H(X, Y|Z) = 0.5[H(X, Y|Z = 0) + H(X, Y|Z = 1)] = 1$, which implies $H(X, Y) = 2 > H(X, Y|Z)$, and so, $X$ and $Y$ are not independent given $Z$.

A2 Note

\[ P(Y = 0) = \frac{1}{2} \left[ P_{Y|X}(0|0) + P_{Y|X}(0|1) \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} + 0 \right] \]
\[ = \frac{1}{4} . \]

Thus, $H(Y) = h(0.25) = 0.811$. Furthermore,

\[ H(X, Y) = H(X) + H(Y|X) \]
\[ = 1 + \frac{1}{2}[H(Y|X = 0) + H(Y|X = 1)] \]
\[ = 1 + \frac{1}{2}[1 + 0] \]
\[ = \frac{3}{2} . \]

Finally, $H(X|Y) = H(X, Y) - H(Y) = 0.689$. 

1
A3  a) Note that $N_1 = n$ occurs when either the first $n - 1$ tosses are heads and the $n$th toss is tails or the first $n - 1$ tosses are tails and the $n$th toss is heads. Thus,

$$P(N_1 = n) = 2 \cdot \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}, \quad n \geq 2,$$

and consequently,

$$H(N_1) = \sum_{n=2}^{\infty} (n - 1) \cdot \left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n = 2,$$

where the final equality follows upon denoting the right side above by $S$ and noting that

$$S = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n = \frac{1}{2} \left[ 1 + \sum_{n=2}^{\infty} (n - 1 + 1) \cdot \left(\frac{1}{2}\right)^{n-1} \right] = \frac{1}{2} \left[ 1 + S + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \right] = 1 + \frac{S}{2}.$$

b) Note that the random variable $Y$ is a one-to-one function of $(X_1, X_2, X_3, X_4, X_5)$. Therefore,

$$H(Y) = H(X_1, X_2, X_3, X_4, X_5) = 5H(X_1) = 5.$$

A4  It is easy to show the bounds in the question directly. These bounds are a special case of more general bounds, which we will derive here.

Let $A_1, ..., A_k$ be a partition of the set $[m] = \{1, ..., m\}$, i.e., $A_i$’s are disjoint and their union is $[m]$. For a set $A$, let $X_A$ denote the collection of random variables $(X_i, i \in A)$. We shall show that

$$H(X_{[m]}) \leq \frac{1}{k-1} \sum_{i=1}^{k} H(X_{A_i^c})$$

and that

$$H(X_{[m]}) \geq \frac{1}{k-1} \sum_{i=1}^{k} H(X_{A_i^c}|X_{A_i}) ,$$

where $A^c = [m] \setminus A$. The inequalities in the question correspond to the case $m = 3$ and the partition $\{\{1\}, \{2\}, \{3\}\}$. 
The upper bound holds since
\[
\frac{1}{k-1} \sum_{i=1}^{k} H\left( X_{A_i^c} \right) = \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j \in A_i^c} H\left( X_j | X_{[j-1] \cap A_i^c} \right)
\]
\[
= \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \mathbb{1}(j \in A_i^c) H\left( X_j | X_{[j-1]} \right)
\]
\[
\geq \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \mathbb{1}(j \in A_i^c) H\left( X_j | X_{[j-1]} \right)
\]
\[
= \frac{1}{k-1} \sum_{j=1}^{m} \sum_{i=1}^{k} \mathbb{1}(j \in A_i^c) H\left( X_j | X_{[j-1]} \right)
\]
\[
= \frac{1}{k-1} \sum_{j=1}^{m} \mathbb{1}\{i \in [k] : j \in A_i^c\} |H\left( X_j | X_{[j-1]} \right)|
\]
\[
= \frac{1}{k-1} \sum_{j=1}^{m} (k-1) H\left( X_j | X_{[j-1]} \right)
\]
\[
= \sum_{j=1}^{m} H\left( X_j | X_{[j-1]} \right)
\]
\[
= H(X_{[m]}),
\]
where we have used the fact that each element \( j \in [m] \) belongs to \((k-1)\) of the sets \( A_i^c \).

The lower bound holds since
\[
\frac{1}{k-1} \sum_{i=1}^{k} H\left( X_{A_i^c} | X_{A_i} \right) = \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j \in A_i^c} H\left( X_j | X_{[j-1]} \cap A_i^c, X_{A_i} \right)
\]
\[
= \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \mathbb{1}(j \in A_i^c) H\left( X_j | X_{[j-1]} \cap A_i^c, X_{A_i} \right)
\]
\[
\leq \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \mathbb{1}(j \in A_i^c) H\left( X_j | X_{[j-1]} \right)
\]
\[
= H(X_{[m]}),
\]
where the final equality follows by repeating the steps in the proof of the upper bound above.

**A** Let \( X \) be distributed uniformly over all subsets of \( \{1, \ldots, n\} \) of size less than or equal to \( np \).

Let \( X_i = \mathbb{1}(i \in X) \), i.e., \( X_i \) is 1 if the random set contains the element \( i \) and 0 otherwise.

Thus, \( X = (X_1, \ldots, X_n) \).

Note that the number of values taken by the random variable
\[
\sum_{i \leq np} \binom{n}{i}.
\]
Thus,

\[ \log \sum_{i \leq np} \binom{n}{i} = H(X) \]

\[ = H(X_1, \ldots, X_n) \]

\[ \leq \sum_{i=1}^{n} H(X_i) \]

\[ = nH(X_1), \]

where in the last step we used the fact that random variables \( X_i \)'s have the same distribution.

To bound \( H(X_1) \), we bound \( P(X_1 = 1) \). To that end, note that for every \( \delta \leq p \)

\[ P(X_1 = 1|X = n\delta) = \frac{\text{number of subsets of } [n] \text{ of size } n\delta \text{ containing 1}}{\text{number of subsets of } [n] \text{ of size } n\delta} \]

\[ = \frac{\binom{n-1}{n\delta-1}}{\binom{n}{n\delta}} \]

\[ = \delta \]

\[ \leq p. \]

Therefore, \( P(X_1 = 1) \leq p \). Since \( h(\delta) \) is increasing in \( \delta \) for \( 0 \leq \delta \leq 1/2 \),

\[ H(X_1) = h(P(X_1 = 1)) \leq h(p), \]

which together with (1) completes the proof.