Reading Assignment

- Read Chapter 8 and Sections 9.1, 9.2, 9.4 of the Cover and Thomas book.

Homework Questions

Only the first 4 questions will be graded.

Q1 Consider a DMC $W : \mathcal{X} \rightarrow \mathcal{Y}$, and let $C(W)$ denote its capacity. We cascade $W$ with an erasure channel with input alphabet $\mathcal{Y}$ and output alphabet $\mathcal{Y} \cup \{e\}$. For each input $y \in \mathcal{Y}$, the erasure channel produces the output $y$ with probability $(1 - \alpha)$ and the output $e$ with probability $\alpha$. Determine the capacity of the combined channel with input $\mathcal{X}$ and output $\mathcal{Y} \cup \{e\}$.

Q2 Consider a Binary Symmetric Channel with crossover probability $\epsilon$. A code of length $n$ and size $2^k$ for this channel is said to be an $(n,k,d)$ code if the minimum number of bits in which two codewords differ is $d$, i.e., any two codewords differ in no less than $d$-bits and there is a pair of codewords that differs in exactly $d$ bits. Consider the minimum distance decoder which decodes a received binary vector to the nearest codeword (in case of clash, a codeword is chosen at random).

a) Find an upper bound for the maximum probability of error for an $(n,k,d)$ code with minimum distance decoder in terms of $n,k,d,$ and $\epsilon$. Your bound should make itself amenable to the next part.

b) Consider a sequence of $(n,nR,n\delta)$ code with the minimum distance decoder, i.e., a code of rate $R$ and minimum distance scaling linearly with the length $n$. Using your bound in the previous part, for a fixed $R > 0$, determine an allowed range for the value of $\delta > 0$ such that the maximum probability of error for this code goes to 0 as $n$ goes to $\infty$.

Q3 Consider two DMCs $W_1 : \mathcal{X} \rightarrow \mathcal{Y}_1$ and $W_2 : \mathcal{X} \rightarrow \mathcal{Y}_2$ with the same input alphabet $\mathcal{X}$; let their capacities be $C_1$ and $C_2$, respectively. Consider a new DMC $\tilde{W} : \mathcal{X} \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$ with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ defined by

$$\tilde{W}(y_1,y_2|x) = W_1(y_1|x)W_2(y_2|x).$$

Let $\tilde{C}$ be the capacity of the DMC $\tilde{W}$.

a) Establish a relationship between $\tilde{C}, C_1,$ and $C_2$.

b) Provide an example where $\tilde{C}$ is strictly larger than both $C_1$ and $C_2$.

Q4 (a) For a DMC $W$, consider a sequence of codes $(f_n, \phi_n)$ length $n$, rate $R$ and with average probability of error $\epsilon_n \rightarrow 0$ in the limit as $n \rightarrow \infty$. For a fixed length $n$, let $M_n$ denote
a random message distributed uniformly over $2^{nR}$ messages, and let $Y_1, ..., Y_n$ denote the corresponding channel outputs. Using Fano’s inequality, show that
\[
R = \lim_{n \to \infty} \frac{1}{n} I(M_n \wedge Y^n).
\]

(b) Using the previous part, show that for codes with feedback of rate $R$ with average probability of error approaching 0,
\[
R = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I(X_i \wedge Y_i|Y_1, ..., Y_{i-1}),
\]
where $X_1, ..., X_n$ denote the random channel inputs corresponding to a uniformly distributed message. Conclude that feedback capacity is no more than $\max P I(P; W)$.

Q5 (Alternative Random Coding Argument.) For a DMC $W: \mathcal{X} \to \mathcal{Y}$, consider a randomly chosen code $C$ of size $N = 2^{nR}$ where the codewords $X_1, ..., X_N$ are chosen i.i.d. uniformly over the typical set $\mathcal{T}_P \subset \mathcal{X}^n$. Consider this random code with the conditional typical decoder described below:

\[
\phi(y) = i \quad \text{if } X_i \text{ is the unique codeword such that } y \in \mathcal{T}_W(X_i); \quad \text{if no such codeword is found, or if more than one such codewords are found, an error is declared.}
\]

Denote by $\epsilon(C)$ the average probability of error for the code $C$. For $R < I(P; W)$, we shall show that the expected average probability of error $E_C[\epsilon(C)] \to 0$ in the limit $n \to \infty$, i.e.,
\[
\lim_{n \to \infty} E_C[\epsilon(C)] = 0. \tag{1}
\]
To that end, for output $Y^n = (Y_1, ..., Y_n)$ of the channel, denote by $\mathcal{E}_i$ the event
\[
\mathcal{E}_i = \{Y^n \in \mathcal{T}_W(X_i)\}, \quad 1 \leq i \leq N.
\]

(a) Show that
\[
\lim_{n \to \infty} P(\mathcal{E}_1|1 \text{ is sent}) = 0.
\]

(b) Let $V$ denote the conditional distribution $P_{X|Y}$ for the joint distribution $P_{XY}(x,y) = P(x)W(y|x)$. Show that for $i \neq 1$,
\[
\mathcal{E}_i = \{X_i \in \mathcal{T}_W(Y^n)\},
\]
and conclude that
\[
P(\mathcal{E}_i|1 \text{ is sent}, X_1 = x) = \frac{|\mathcal{T}_W(x)|}{|\mathcal{T}_P|}.
\]

(c) Finally, conclude (1) from the two observations above and proceeding as in the random coding proofs done in the class.

\[1\text{If you want to be rigorous, you need to show that the limit on the right exists. Otherwise, you can just assume this fact.}\]
Consider jointly Gaussian random variables \((Z_1, Z_2)\) with zero mean and covariance matrix

\[
K = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.
\]

Consider a channel \(W\) with input \(\mathbb{R}\) and output \(\mathbb{R}^2\) such that for every input \(x \in \mathbb{R}\) the channel output \((Y_1, Y_2)\) is given by

\[
Y_i = x + Z_i, \quad i = 1, 2.
\]

(a) Following the steps of the proof for AWGN channel capacity done in the class, determine the capacity \(C_P(W)\) for codes with codewords \(x \in \mathcal{X}^n\) satisfying the average power constraint

\[
\frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P.
\]

(b) Find the capacity \(C(W)\) for the cases (i) \(\rho = 1\), (ii) \(\rho = 0\), and (iii) \(\rho = -1\).