Reading Assignment


Homework Questions

Q1 a) A coin shows heads with (unknown) probability \( p \). When it was tossed 1000 times, 400 heads were observed. Find the maximum likelihood estimate of \( p \), i.e., find \( p \) that maximizes the probability of 400 heads showing up in 1000 tosses.

b) Suppose a sequence \( x = (x_1, ..., x_n) \) is observed in \( n \) independent draws from a (unknown) pmf \( P \). Find the maximum likelihood estimate \( \hat{P} \) of \( P \) defined as

\[
\hat{P}(x) = \arg\max_{P} P^n(x).
\]

Q2 For the binary alphabet \( \mathcal{X} = \{0, 1\} \), the type of a sequence \( x = (x_1, ..., x_n) \in \mathcal{X}^n \) is determined by its Hamming weight \( w(x) \) = number of 1’s in \( x \). Specifically, a sequence \( x \) is of type \( p \in [0, 1] \cap \mathbb{Q} \) if it has weight \( np \).

Prove the following bounds, which are a special case of the general results derived in class applied to the special case of binary alphabet. (Don’t just cite the results derived in the class; give a complete proof).

a) Consider a pmf \( P \) on \( \mathcal{X} \) such that \( P(1) = p \). Then, for every sequence \( x \in \mathcal{X}^n \) of type \( q \),

\[
P^n(x) = 2^{-n[D(q||p)+h(q)]},
\]

where \( h(q) = -q \log q - (1-q) \log (1-q) \) is the binary entropy of \( q \) and the binary divergence function \( D(q||p) \) is given by \( q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p} \).

b) Denote by \( \mathcal{T}_q \) the set of all \( n \)-length binary sequences of type \( q \). Show that

\[
\frac{2^{nh(q)}}{(n+1)} \leq |\mathcal{T}_q| \leq 2^{nh(q)}.
\]

c) How many types \( q \) are there such that the number of sequences of type \( q \) does not grow exponentially in \( n \)?

Q3 For a pmf \( P \) on a finite alphabet \( \mathcal{X} \), show that

\[
P^n \left( \{ x : D(P_x||P) > \epsilon \} \right) \leq (n+1)^{|\mathcal{X}|} 2^{-n\epsilon}.
\]

Q4 Consider a DMS with a finite alphabet \( \mathcal{X} \) and common distribution \( P \). For \( n \geq 1 \), show that for every type \( Q \) and every \( x \in \mathcal{T}_Q \)

\[
2^{-nH(Q)} \leq P \left( X^n = x | X^n \in \mathcal{T}_Q \right) \leq (n+1)^{|\mathcal{X}|} 2^{-nH(Q)}.
\]