E2 201: Information Theory (2015)
Homework 1
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Reading Assignment

• Verify that Lemma 1 and Lemma 2 proved in the class hold even when the discrete alphabet \( \mathcal{X} \) is not finite.


Homework Questions

Instructions: You can either answer Q1-Q4 or the (*) question

Consider three sources \( X_1, X_2 \) and \( X_3 \), all with a common alphabet \( \mathcal{X} = \{0, 1\}^{10} \). The first source produces all sequences with equal probability, i.e.,

\[
P_{X_1}(x) = 2^{-10}, \quad \text{for all } x \in \{0, 1\}^{10}.
\]

The second source only produces sequences starting with 00000, each with equal probability, i.e.,

\[
P_{X_2}(x_1, \ldots, x_{10}) =
\begin{cases}
2^{-5}, & \text{if } x_1 = x_2 = x_3 = x_4 = x_5 = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( X_3 \) be a uniform mixture of sources \( X_1 \) and \( X_2 \), i.e.,

\[
P_X(x) = \frac{1}{2}P_{X_1}(x) + \frac{1}{2}P_{X_2}(x), \quad \text{for all } x \in \mathcal{X}.
\]

Q1 Determine the entropies \( H(X_1), H(X_2), \) and \( H(X) \).

Q2 For \( \epsilon = 0.01 \), determine \( L_\epsilon(X_1) \) and \( L_\epsilon(X_2) \).

Q3 For \( \epsilon = 0.01 \), using Lemma 1 and Lemma 2 proved in the class find bounds for \( L_\epsilon(X_3) \). Does \( L_\epsilon(X_3) \) equal \( H(X_3) \)?

Q4 Denote by \( X_1^n, X_2^n \), and \( X_3^n \) discrete memoryless sources (DMSs) with common distributions \( P_{X_1}, P_{X_2}, \) and \( P_{X_3} \), respectively. Let \( \epsilon = 0.01 \). For each of the DMS above, determine \( R^*_\epsilon \), the least \( \epsilon \)-achievable rate of a source code.
For a pmf $P$, the Rényi entropy $P$ of order $\alpha > 0, \alpha \neq 1$, is defined as

$$H_\alpha(P) \overset{\text{def}}{=} \frac{1}{1-\alpha} \log \sum_{x \in X} P(x)^\alpha.$$ 

Consider a source $X$ with pmf $P$. Let $0 < \epsilon < 1$. Using Lemma 1 and Lemma 2 from the class, show that for every $0 < \alpha < 1$

$$L_\epsilon(X) \leq H_\alpha(P) + \frac{1}{1-\alpha} \log \frac{1}{\epsilon} + 1.$$ 

Furthermore, for every $\beta > 1$ and $0 < \delta < 1 - \epsilon$,

$$L_\epsilon(X) \geq H_\beta(P) - \frac{1}{\beta-1} \log \frac{1}{\delta} - \log \frac{1}{1-\epsilon-\delta}.$$ 

[Hint: Verify the assumptions of Lemma 1 for $\lambda = H_\alpha(P) + (1-\alpha)^{-1} \log(1/\epsilon)$ and of Lemma 2 for $\lambda' = H_\beta(P) - (\beta - 1)^{-1} \log(1/\delta).$]