(i) $p = 0.5, q = 0.51$

$$D(P||Q) = \frac{1}{2} \log \frac{0.5}{0.51} + \frac{1}{2} \log \frac{0.5}{0.49}$$

$$= 2.886 \times 10^{-4} \quad (1)$$

We know from Stein’s lemma that for large values of $n$, $\beta_n(P, Q) \approx 2^{-nD(P||Q)}$. Thus for $n = 10^6, 0.01$, and using Stein’s lemma $\beta_{0.01}(P^n, Q^n) \approx 2^{-288.6}$.

(ii)

$$Var \left( \log \frac{p(x^n)}{q(x^n)} \right) = E \left[ \left( \log \frac{p}{q} \right)^2 \right] - D(P||Q)^2$$

$$= \frac{1}{2} \left( \log \frac{0.5}{0.51} \right)^2 + \frac{1}{2} \left( \log \frac{0.5}{0.49} \right)^2 - (2.886 \times 10^{-4})^2 \quad \text{(from (1))}$$

$$= 8.3277 \times 10^{-4}$$

From Chebyshev’s inequality we have,

$$P \left( \log \frac{p(x^n)}{q(x^n)} \geq nD(P||Q) - \sqrt{nVar \left( \log \frac{p}{q} \right) \epsilon} \right) \geq 1 - \epsilon.$$ 

Thus, from Little-Big lemmas we have

$$\lambda = nD(P||Q) - \sqrt{nVar \left( \log \frac{p}{q} \right) \epsilon}.$$ 

For $n = 10^6, \epsilon = 0.5$ and substituting for divergence and variance from above

$$\lambda = 0.0225.$$ 

Similarly, from the other side of Chebyshev’s inequality we have,

$$P \left( \log \frac{p(x^n)}{q(x^n)} \leq nD(P||Q) + \sqrt{nVar \left( \log \frac{p}{q} \right) \delta} \right) \geq 1 - \delta.$$ 

$$\Rightarrow \lambda' = nD(P||Q) + \sqrt{nVar \left( \log \frac{p}{q} \right) \delta}.$$
Choosing $\delta = \epsilon = 0.01$, and using values for the other parameters as above, we get $\lambda' = 577.18$. From the two lemmas, $\lambda$ and $\lambda'$,

$$2^{-577.18} \leq \beta_{0.01}(P, Q) \leq 0.9845$$

As we see, the bounds derived using Chebyshev’s inequality is very weak and compares badly with the approximation provided by Stein’s lemma.