Reading Assignment

- Read Chapter 7 of the Cover and Thomas book (you may skip 7.11) and Chapter 6 of the Csiszár and Körner book till Theorem 6.10.

Homework Questions

Q1 Compute the capacity for the following channels and determine the capacity achieving input distribution:
   
   (a) *Additive noise channel.* A DMC with \( X = Y = \{0, 1, \ldots, m\} \) such that for every input \( x \in X \) the output \( Y = x + Z \mod (m + 1) \), where the random variable \( Z \) takes the value 0 with probability \( 1 - \delta \) and all other values with equal probability \( \delta/m \).

   (b) *The Z-channel.* A DMC with \( X = Y = \{0, 1\} \) and stochastic matrix \( W \) given by:

   \[
   W(y|x) = \begin{cases} 
   1, & x = 0, y = 0 \\
   0, & x = 0, y = 1 \\
   1/2, & x = 1, y = 0, 1.
   \end{cases}
   \]

Q2 In the hope of increasing the capacity of a channel \((X, W, Y)\), an M.Tech. student taking the DSP course applied a random transform \( T : Y \rightarrow Z \) to the output \( Y \) of the channel \( W \).

   (a) Show that the student will fail to increase the capacity.

   (b) Under what conditions does he not strictly decrease the capacity?

Q3 *Cascade of binary symmetric channels.* Consider a cascade of \( n \) identical binary symmetric channels (BSCs) with crossover probability \( \delta \), \( 0 < \delta < 1 \). No encoding or decoding occurs at the intermediate points of the cascade.

   (a) Show that this cascade is equivalent to a single BSC with crossover probability equal to \( \frac{1}{2} (1 - (1 - 2\delta)^n) \).

   (b) Let the \( \{0, 1\}\)-valued r.v. \( X \) represent the input to the first BSC in the cascade, and let the \( \{0, 1\}\)-valued r.v. \( Y_n \) represent the output of the \( n^{th} \) BSC in the cascade, \( n = 1, 2, \ldots \).

   Show that for every choice of the pmf \( P_X \) of the r.v. \( X \), it holds that \( \lim I(X \wedge Y_n) = 0 \). Thus, conclude that the capacity of the DMC defined by the cascade tends to 0 as \( n \) tends to \( \infty \).

   (c) What would be the value of the capacity in the limit in part (b) above if suitable encoding and decoding were allowed at each intermediate point of the cascade?

Q4 *Sum of channels.* The sum of two DMCs \((X_1, W_1, Y_1)\) and \((X_2, W_2, Y_2)\) with

   \[
   X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset
   \]
is the DMC $(\mathcal{X}_1 \cup \mathcal{X}_2, W_1 \oplus W_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$, where the stochastic matrix $W_1 \oplus W_2$ is defined by:

$$(W_1 \oplus W_2)(y|x) \overset{\Delta}{=} \begin{cases} W_1(y|x), & \text{if } x \in \mathcal{X}_1, y \in \mathcal{Y}_1 \\ W_2(y|x), & \text{if } x \in \mathcal{X}_2, y \in \mathcal{Y}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Show that the capacity of $W_1 \oplus W_2$ equals $\log(2^{C_1} + 2^{C_2})$, where $C_i$ is the capacity of the DMC $W_i$, $i = 1, 2$.

Q5 Product of channels. The product of two DMCs $(\mathcal{X}_1, W_1, \mathcal{Y}_1)$ and $(\mathcal{X}_2, W_2, \mathcal{Y}_2)$ is the DMC $(\mathcal{X}_1 \times \mathcal{X}_2, W_1 \times W_2, \mathcal{Y}_1 \times \mathcal{Y}_2)$, where the stochastic matrix $W_1 \times W_2$ is defined by:

$$(W_1 \times W_2)(y_1, y_2|x_1, x_2) \overset{\text{def}}{=} W_1(y_1|x_1)W_2(y_2|x_2).$$

Show that the capacity of $W_1 \times W_2$ equals the sum of the capacities of $W_1$ and $W_2$.

Q6 Consider a DMC $(\mathcal{X}, W, \mathcal{Y})$, and let $C(W)$ denote its capacity. We cascade $W$ with an erasure channel with input alphabet $\mathcal{Y}$ and output alphabet $\mathcal{Y} \cup \{e\}$. For each input $y \in \mathcal{Y}$, the erasure channel produces the output $y$ with probability $(1 - \alpha)$ and the output $e$ with probability $\alpha$.

Determine the capacity of the combined channel with input $\mathcal{X}$ and output $\mathcal{Y} \cup \{e\}$.

Q7 Consider the setup of previous channel with $W$ being a BSC with cross over probability $\delta$.

(a) Modify the maximal code construction for BSC done in the class to directly show that the capacity you computed in the last part is an achievable rate.

(b) Modify the the sphere packing bound proof for BSC to establish the strong converse for the combined channel.

Q8 Consider two DMCs $(\mathcal{X}, W_1, \mathcal{Y}_1)$ and $(\mathcal{X}, W_2, \mathcal{Y}_2)$ with the same input alphabet $\mathcal{X}$; let their capacities be $C_1$ and $C_2$, respectively. Consider a new DMC $(\mathcal{X}, \tilde{W}, \mathcal{Y}_1 \times \mathcal{Y}_2)$ with the input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ defined by

$$\tilde{W}(y_1, y_2|x) = W_1(y_1|x)W_2(y_2|x).$$

Let $\tilde{C}$ be the capacity of the DMC $\tilde{W}$.

(a) Establish a relationship between $\tilde{C}$, $C_1$, and $C_2$.

(b) Provide an example where $\tilde{C}$ is strictly larger than both $C_1$ and $C_2$. 