Linear filtering methods for fixed rate quantisation with noisy symmetric error channels

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Abstract: This study considers linear filtering methods for minimising the end-to-end average distortion of a fixed-rate source quantisation system. For the source encoder, both scalar and vector quantisation are considered. The codebook index output by the encoder is transmitted over a noisy discrete memoryless channel whose statistics could be unknown at the transmitter. At the receiver, the code vector corresponding to the received index is passed through a linear receive filter, whose output is an estimate of the source instantiation. Under this setup, an approximate expression for the average weighted mean-square error (WMSE) between the source instantiation and the reconstructed vector at the receiver is derived using high-resolution quantisation theory. Also, a closed-form expression for the linear receive filter that minimises the approximate average WMSE is derived. The generality of framework developed is further demonstrated by theoretically analysing the performance of other adaptation techniques that can be employed when the channel statistics are available at the transmitter also, such as joint transmit–receive linear filtering and codebook scaling. Monte Carlo simulation results validate the theoretical expressions, and illustrate the improvement in the average distortion that can be obtained using linear filtering techniques.

1 Introduction

Fixed-rate quantisation (FRQ) schemes such as scalar quantisation (SQ) and vector quantisation (VQ) are used in lossy source compression for two main reasons: simple off-the-shelf algorithms are available for designing locally optimum code books that minimise the average distortion, and the performance of FRQ can be analysed using high-rate quantisation theory [1, 2]. However, when the index output by the encoder is transmitted over a noisy channel, FRQ-based source compression can be very sensitive to the errors introduced by the channel, resulting in a significant degradation in the average distortion performance [3, 4]. On the other hand, in many applications, the channel statistics are unknown at the time of encoding. This precludes the use of techniques such as channel-optimised VQ (COVQ) [5], scaling the codebook [6, 7], index assignment (IA) [8, 9] or a combined source and channel adaptive quantisation [10]. For example, in recording media such as the compact disc, or the reverse-link feedback of channel state information in multiple antenna systems, the channel statistics are not known at the time of recording/transmission. This motivates one to consider techniques for reducing the average distortion that can be implemented solely at the receiver. In this paper, we design, and analyse, the efficacy of linear filtering techniques for mitigating the excess distortion introduced when the codebook index output by an FRQ encoder is transmitted over a discrete memoryless channel (DMC).

High-rate analysis of FRQ for noiseless channels has been studied by many authors [3, 11, 12]. There is also extensive literature on the performance and optimisation of VQ for noisy channels, for example, with optimum IA [8, 4], with COVQ [5, 4, 13] and with soft decision VQ [14, 15]. The high-rate analysis has also been extended to the noisy symmetric error channel [16–18]. Other related literature includes the scaled codebook approach [6, 7], where a real-valued scaling factor is applied at the transmitter and receiver to reduce the overall distortion.

Aforementioned techniques for improving the distortion performance of FRQs with noisy channels suffer from two main drawbacks. First, they require knowledge of the channel statistics at the transmitter, which may not always be available. Second, they are computationally intensive to optimise (e.g. COVQ, IA or the scaled codebook) when the channel statistics change over time. In particular, there is little past work on the design and optimisation of receiver-only techniques for FRQ. Linear filtering, the focus of our study in this paper, is a simple, yet effective, adaptation technique that can help in reducing the average distortion performance of FRQ-based source compression schemes when the channel is noisy. Our contributions are as follows:

- We derive analytical expressions that approximate the average weighted mean-squared error (WMSE) distortion of source coding for noisy channels when an LRF is applied after the source decoder, with random IA (defined in Section 2). By setting the LRF to the identity matrix, we obtain the approximate WMSE performance without filtering as a special case of the analysis (see Section 3).
Lloyd-Max algorithm [20] is used to design codebooks that represent zero mean, variance C function (pdf) indices to symbols that are sent over the channel is chosen at the source decoder. In this work, we consider random noisy DMC, and is received as a possibly different index that is, there are roughly \( N^c \). We use the notation \( N \in \mathbb{N} \), where \( N \) is the number of quantisation levels per dimension.

Owing to RIA, the channel is equivalent to a symmetric error channel (SEC) [16, 17]. Here, the transition probability of receiving index \( j \) when index \( i \) is transmitted, is given by \( P_{ij} = \epsilon_n + (1 - N \epsilon_n) \delta(i, j) \), where \( \delta(i, j) \) is the Kronecker delta function. Typically, \( P_{ij} \geq P_{ji} \), \( i \neq j \), and hence, the index error rate \( \epsilon_n \) satisfies \( 0 \leq \epsilon_n \leq 1/(N - 1) \). For SQ, \( N = N^c \).

At the source decoder, upon receiving \( j \), the decoder first outputs the corresponding codebook entry \( y = \hat{x}_j \), which is then multiplied by a linear receive filter \( \mathbf{R} \) to obtain \( \tilde{y} \) as an estimate of \( x \). Thus, the end-to-end distortion in the source vector \( x \in \mathcal{R}_i \), is \( d(x, \mathbf{R}y) \). The average distortion is given by

\[
J = E \left[ (x - \tilde{y})^T W (x - \tilde{y}) \right]
\]

\[
= \sum_{j=1}^{N} P_{ij} \int_{\mathcal{R}_j} (x - \tilde{R}_j)^T W (x - \tilde{R}_j) f_s(x) \, dx
\]

where the expectation is taken over both the source distribution and the channel transition probabilities. Hence, our goal in this paper is to analyse (1) and design the LRF, \( \tilde{R} \), to minimise the average WMSE distortion. To this end, we use high-rate quantisation theory to derive an approximate but closed-form expression for the WMSE, and then find the LRF that minimises the approximate WMSE. The next section presents the derivation of the LRF.

\section{Problem setup}

We consider a random \( n \)-dimensional source vector \( x \) with zero mean, variance \( \sigma^2 \) and continuous probability density function (pdf) \( f_d(x) \) over a compact support \( \mathcal{D}_c \subset \mathbb{R}^n \). The source encoder maps \( x \) to the closest vector \( y \) in a codebook \( \mathcal{C} \) of cardinality \( N \), with respect to the WMSE distortion \( d(x, y) = (x - y)^T W (x - y) \), where \( W \) is an \( n \times n \) symmetric positive definite matrix. Also, for SQ, the matrix \( W \) is assumed to be diagonal. For both SQ and VQ, the Lloyd-Max algorithm [20] is used to design codebooks that are source-optimised for a noiseless channel. We denote the source-optimised codebook by the set \( \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N\} \), and the corresponding point density function \( \lambda(x) \) that is, there are roughly \( N \lambda(x) \) code points in a small volume \( dx \) containing \( x \in \mathcal{D}_c \) (see also Appendix). When a linear transmit filter \( \mathbf{T} \) is employed, the encoder uses the transformed codebook \( \mathcal{C} = \{\mathbf{T}\hat{x}_1, \mathbf{T}\hat{x}_2, \ldots, \mathbf{T}\hat{x}_N\} \) for nearest-neighbour-based quantisation. That is, the encoder outputs index \( i \), whenever \( x \in \mathcal{R}_i \triangleq \{x: d(x, \mathbf{T}\hat{x}_j) \leq d(x, \mathbf{T}\hat{x}_i) \text{ for } 1 \leq j \leq N\} \). We use the notation \( \mathcal{R}_i \) to represent \( \mathcal{R}_i \), that is, without transmit filtering.

The index \( i \) output by the source encoder is sent over a noisy DMC, and is received as a possibly different index \( j \) at the source decoder. In this work, we consider random index assignment (RIA), where the assignment of codebook indices to symbols that are sent over the channel is chosen uniformly at random from all possible IAs. This is consistent with the assumption that the transmitter is not cognizant of the channel statistics, and is hence unable to optimise the IA. A detailed justification for the RIA assumption can be found in [18]. Note that, the theory developed here can also be extended to analyse the distortion for specific index assignments using a convex combination of the distortion with optimum IA and random IA, as in [21].

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\]

where the expectation is taken over both the source distribution and the channel transition probabilities. Hence, our goal in this paper is to analyse (1) and design the LRF, \( \tilde{R} \), to minimise the average WMSE distortion. To this end, we use high-rate quantisation theory to derive an approximate but closed-form expression for the WMSE, and then find the LRF that minimises the approximate WMSE. The next section presents the derivation of the LRF.

\section{Linear receive-only filtering}

In this section, we consider receive-only filtering, that is, \( \mathbf{T} = \mathbf{I} \). Let \( x \in \mathcal{R}_i \) be the source instantiation and \( y \) be the corresponding codeword at the receiver, which could be different from \( \hat{x}_i \) because of channel errors. Then, the vector \( y \) can be written as

\[
y = x + n
\]

where \( n \) is an additive noise vector. Owing to the structure in the index transition probability matrix, we consider two cases for \( y \) separately: \( y = \hat{x}_i \), when the correct index is received, and \( y = \hat{y} \), when an incorrect index is received. Hence, when the channel makes no error, \( n = \hat{x}_i - x \). When the channel does make an error, \( n = \hat{y} - x \), where \( \hat{y} \) is chosen uniformly among all other codewords \( \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{i-1}, \hat{x}_{i+1}, \ldots, \hat{x}_N\} \). Equivalently, \( n \) is distributed as

\[
f_n(n) = \begin{cases} 
(\hat{x}_i - x), & \text{with probability } 1 - (N - 1)e_n \\
(\hat{y} - x), & \text{with probability } (N - 1)e_n
\end{cases}
\]

\subsection{Linear receive filter for VQ}

The first main result of this paper is stated as the following theorem. In stating the theorem, the quantity \( E_{\tilde{y}^VQ} \) is defined as the minimum WMSE obtained under the high-rate approximations, and hence, (5) is presented with an equality.

\textbf{Theorem 1:} The LRF that minimises the approximate average WMSE distortion in (1) when the index output by a VQ encoder is transmitted over an SEC with index error rate \( e_n \)
is given by

$$ R_{\text{opt}} = \tilde{S}_{xx} \left[ \frac{N \varepsilon_N}{1 - N \varepsilon_N} S_\lambda + \tilde{S}_{xx} \right]^{-1} \tag{4} $$

where

$$ \tilde{S}_{xx} \triangleq S_{xx} - \Theta, \quad S_{xx} \triangleq E[xx^T], \quad \Theta \triangleq \Phi_0 \Gamma_n N^{-(n-2)/n}, \quad \Phi_0 \triangleq \frac{N^{(n-2)/n} |W|^{(1/n)}}{n + 2} W^{-1} \quad \Gamma_n \triangleq \left[ \int_{D_x} f_n^{(n)(x+2/n)}(x) \, dx \right]^{(n+2)/n} \quad \text{and} $$

$$ S_\lambda \triangleq \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_j \tilde{x}_i^T \approx \int_{D_x} y \, \mu_\lambda^T \lambda(y) \, dy \quad \text{using the distribution of } n \text{ in (3)}, \quad E[nn^T] \text{ can be written as} $$

$$ E[nn^T] = \left( 1 - (N - 1) \varepsilon_N \right) E[xx^T] + (N - 1) \varepsilon_N E[yy^T] \tag{6} $$

The expectation in the first term above is approximated as

$$ E\left[ x(\tilde{x}_i - x)^T \right] = \sum_{j=1}^{N} \int_{E_j} x(\tilde{x}_j - x)^T f_e(x) \, dx \tag{7} $$

$$ \approx - \sum_{j=1}^{N} f_e(x) \int_{E_j} \tilde{x}_j \, e(\tilde{x}_j + e) \lambda \, de = - \Theta \tag{8} $$

where $e \triangleq (x - \tilde{x})$ and $E_j \triangleq \{ ee + \hat{x}_j \in R_i \}$. The above is obtained using the approximations that the polytope generating the Voronoi regions is geometrically centered about the origin [2] and the expression for $E[ee^T]$ from (39) in Appendix 2.

Similarly, $E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right]$ can be shown to be given by (see (9)).

The fact that $E[x] = 0$ and the approximation in (8) has been used to obtain (9). Substituting (8) and (9) in (6), we have $E[nn^T] = - (1 - N \varepsilon_N) \Theta + N \varepsilon_N S_{xx}$, and defining $S_{xx} \triangleq S_{xx} - \Theta$, we obtain

$$ S_{xy} = (1 - N \varepsilon_N) \tilde{S}_{xx} \tag{10} $$

Now, $S_{yy} = E[(x + n)y] = E[xy^T] + E[yn^T] + E[nn^T]$ and note that the first two terms are as derived above. Using (3), $S_{nn} \triangleq E[nn^T]$ is given by

$$ S_{nn} = (1 - (N - 1) \varepsilon_N) E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right] $$

$$ + (N - 1) \varepsilon_N E\left[ (\tilde{x}_i - x)(\tilde{x}_i - x)^T \right] \tag{11} $$

with $\tilde{x}_j$ being the code vector corresponding to the source instantiation $x$ and with $\tilde{x}_j$ being chosen uniformly among all other codewords $\{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \ldots, \tilde{x}_N \}$ because of the structure of the index transition probability matrix. The expectation in the first term is given by (39) in Appendix 2 as $E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right] = \Theta$. To compute the expectation in the second term, note that $E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right] = \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{D_x} E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right] f_e(x) \, dx$. Now, if $x \in \mathbb{R}_i$, we have

$$ E\left[ (\tilde{x}_j - x)(\tilde{x}_i - x)^T \right] $$

$$ = \frac{1}{N - 1} \left( \sum_{j=1}^{N} (\tilde{x}_j - x)(\tilde{x}_i - x)^T - (\tilde{x}_i - x)(\tilde{x}_i - x)^T \right) $$

$$ \approx \frac{N}{N - 1} (S_{xx} - x \mu_\lambda^T - \mu_\lambda x^T + xx^T) $$

$$ - \frac{1}{N - 1} (\tilde{x}_i - x)(\tilde{x}_i - x)^T \tag{12} $$

where

$$ \mu_\lambda \triangleq \frac{1}{N} \sum_{j=1}^{N} \tilde{x}_i \approx \int_{D_x} y \lambda(y) \, dy \quad \text{and} \quad S_\lambda \triangleq \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_j \tilde{x}_i^T \approx \int_{D_x} y \lambda(y) \, dy \quad \text{using the distribution of } n \text{ in (3)} \text{, } E[nn^T] \text{ can be written as} $$

$$ R_{opt} = \tilde{S}_{xx} \left[ \frac{N \varepsilon_N}{1 - N \varepsilon_N} S_\lambda + \tilde{S}_{xx} \right]^{-1} \tag{13} $$

$$ \text{IET Signal Process.}, \text{pp. 1–9} $$


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from the Monte Carlo approximation [16]. Quantities $\mu_\lambda$ and $S_\lambda$ can be computed in closed-form given the point density of the codebook. For example, for an $n$-dimensional i.i.d. standard Gaussian source with the source-optimised point density given by (33) in Appendix 1, $\mu_\lambda = 0$ and $S_\lambda = ((n + 2)/n)I$. Hence

$$E[(\hat{y} - y)(\hat{y} - y)^T] \approx \frac{N}{N - 1} \{S_\lambda + S_{xx}\} - \frac{1}{N - 1} \Theta$$

(13)

Again, the fact that $E[x] = 0$ and (39) in Appendix 2 have been used to obtain the above expression. Substituting the above into (11) and using the result to evaluate $S_{yy}$, we obtain

$$S_{yy} = (1 - N\bar{e}_N)\hat{S}_{xx} + N\bar{e}_NS_\lambda$$

(14)

With $S_{xy}$ and $S_{yx}$ in hand, the LRF that minimises the approximate WMSE can be obtained as [22]: $R_{\text{opt}} = S_{xy}S_{yx}^{-1}$. This leads to the expression in (4). Finally, substituting $\hat{R} = R_{\text{opt}}$ in (1) and simplifying yields the following expression for the approximate WMSE:

$$E_{d,\text{VQ}}^\text{conv} = \text{tr} \left[ W \left( S_{xx} - S_{xy}R_{\text{opt}}^T - R_{\text{opt}}S_{yx} + R_{\text{opt}}S_{xy}R_{\text{opt}} \right) \right]$$

$$= \text{tr} \left[ W \left( S_{xx} - (1 - N\bar{e}_N)R_{\text{opt}}\hat{S}_{xx} \right) \right]$$

(15)

The expression in (5) follows by substituting for $R_{\text{opt}}$ from (4) in the above, completing the proof.

### 3.1.1 Comparison with no filtering

The approximate expected WMSE without the LRF is obtained by substituting $\hat{R} = I$ in (1) and simplifying, as

$$E_{d,\text{VQ}}^\text{conv} = \text{tr} \left( W \left( S_{xx} - (1 - N\bar{e}_N)\hat{S}_{xx} + N\bar{e}_N S_\lambda \right) \right)$$

(16)

The above equation is interesting, as it provides an accurate and easy-to-evaluate expression for the expected WMSE of VQ with no filtering and random IA, for a noisy DMC. With some manipulation, it can be shown to be equivalent to the expected distortion expression in the literature [17] (see Theorem 1), and hence, it can be viewed as an alternative and simpler derivation of that result. Now, substituting for $E_{d,\text{VQ}}^\text{conv}$ from (15), the reduction in the WMSE distortion because of the LRF is approximately given by

$$E_{d,\text{VQ}}^\text{conv} - E_{d,\text{VQ}}^{\text{R}_{\text{opt}}} = \text{tr} \left( W \left[ (1 - N\bar{e}_N)R_{\text{opt}}\hat{S}_{xx} - (1 - N\bar{e}_N)\hat{S}_{xx} + N\bar{e}_N S_\lambda \right] \right)$$

(17)

Substituting for $R_{\text{opt}}$ from (4) and replacing $\hat{S}_{xx}$ with

$$\left( \hat{S}_{xx} + \frac{N\bar{e}_N}{1 - N\bar{e}_N} S_\lambda \right) - \frac{N\bar{e}_N}{1 - N\bar{e}_N} S_\lambda$$

for the term marked $a$ in (17), since $\hat{S}_{xx}$ is a symmetric matrix, we obtain

$$E_{d,\text{VQ}}^\text{conv} - E_{d,\text{VQ}}^{\text{R}_{\text{opt}}} = \text{tr} \left( W \left[ N\bar{e}_N (I - \hat{S}_{xx} \left( \frac{N\bar{e}_N}{1 - N\bar{e}_N} S_\lambda \right)^{-1} \right) \right] \right)$$

(18)

Applying the same substitution for the term marked $b$ in (18), we obtain

$$E_{d,\text{VQ}}^\text{conv} - E_{d,\text{VQ}}^{\text{R}_{\text{opt}}} = \text{tr} \left( W \left[ \frac{N\bar{e}_N^2}{1 - N\bar{e}_N^2} S_\lambda \left( \hat{S}_{xx} + \frac{N\bar{e}_N}{1 - N\bar{e}_N} S_\lambda \right)^{-1} \right] \right)$$

(19)

Since $\hat{S}_{xx} = S_{xx} - \Theta$ and $\Theta$ decreases as $N^{-2/(n)}$, $\hat{S}_{xx}$ is positive definite for sufficiently large $N$. Owing to the positive definiteness of $S_\lambda$, the matrix marked $c$ in (19) is positive definite, and hence, the LRF offers a positive improvement performance over no filtering for all channel conditions.

### 3.2 Linear receive filter for SQ

In this section, we derive the LRF that minimises the approximate MSE for SQ of a vector source, and characterise the improvement in the performance from receive filtering. In practice, it is typical to use SQ when the source dimensions are independent and the distortion function is separable, for example, when $W$ is a non-negative diagonal matrix. Hence, in this section, attention is restricted to independent random variables in each of the dimensions of the vector source and the weighting matrix $W$ is set as the identity matrix.

Let $N_s$ be the number of quantisation levels per dimension for SQ, so that $N = N_s^n$ is the size of the overall $n$-dimensional codebook. Within a given dimension, the probability that a codebook index $i$ is incorrectly received as an index $j \neq i$ is $N_s^{(n-1)} \bar{e}_N$. This is because the $n$-dimensional codebook has $N_s^{(n-1)}$ indices with $j$ as the component of the index in the given dimension. Owing to this, we obtain the following marginal index transition probability matrix for each dimension: $P^{(\text{SQ})}_{ij} \triangleq N_s^{(n-1)} \bar{e}_N + (1 - N_s^{(n-1)} \bar{e}_N) \delta(i,j)$, where $1 \leq i \leq N_s$ and $1 \leq j \leq N_s$.

In the previous section, it is interesting to note that the equivalent pairwise index error rate of SQ, $e_{N_s} \triangleq N_s^{(n-1)} \bar{e}_N$, satisfies $N_s e_{N_s} = N\bar{e}_N$. Now, the total distortion with SQ is simply the sum of distortions incurred in each of the dimensions. Hence, the LRF that minimises the approximate MSE in the $i$th dimension, which is a scaling factor denoted by $r_{opt}(i)$, can be obtained by setting $n = 1$ in (4) and substituting the above index transition probability, as follows

$$r_{opt}(i) = \left( \sigma_i^2 - \Gamma_i N_s^{n-2} \right) \left[ \frac{N_s \bar{e}_N \sigma_i^2 + \sigma_i^2 - \Gamma_i N_s^{n-2} \bar{e}_N}{1 - N_s \bar{e}_N \sigma_i^2 + \sigma_i^2 - \Gamma_i N_s^{n-2} \bar{e}_N} \right]^{-1}$$

(20)
where $\sigma_n^2$ and $\sigma_l^2$ are the $i$th diagonal components of $S_{xx}$ and $S_l$, respectively, and

$$
\Gamma_i \triangleq \left[ \int_{D_i} f_{\tilde{x}_i}^{(1/3)}(x) \, dx \right]^3
$$

Here, $f_{\tilde{x}_i}(x)$ is the pdf and $D_i$ is the domain, of the $i$th component of $x$. Using (20) and (15), the approximate expected MSE after the LRF for SQ can be obtained as

$$
E_{d, SQ}^R = \sum_{i=1}^{N} \left( 1 - N E_N \right) \left( \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right)^{-1} \left[ \frac{N E_N}{1 - N E_N} \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right]^{-1}
$$

(21)

### 3.2.1 Comparison with no filtering:

Without receive filtering, we can obtain an expression for the approximate expected distortion of SQ for noisy channels by substituting $n = 1$ in (16) and simplifying, to obtain

$$
E_{d, SQ}^{\text{conv}} = \sum_{i=1}^{N} \left( 1 - N E_N \right) \left( \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right)^{-1} \left[ \frac{N E_N}{1 - N E_N} \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right]^{-1}
$$

(22)

The approximate reduction in the MSE distortion because of the LRF can be obtained from (19) as

$$
E_{d, SQ}^{\text{conv}} - E_{d, SQ}^R
\begin{align*}
&= \sum_{i=1}^{N} \left( 1 - N E_N \right) \left( \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right)^{-1} \\
&\quad \times \left[ \frac{N E_N}{1 - N E_N} \sigma_l^2 + \frac{\sigma_n^2}{1 - N E_N} - \Gamma_i N_l^{-2} \right]^{-1}
\end{align*}
$$

(23)

and the terms in the summation above are all positive for reasonably large $N$. Hence, for high-rate quantisation, the LRF offers a lower MSE distortion compared to the no filtering case. It is interesting to compare the above expressions with the average distortion for VQ obtained in the previous subsection. The simulation results in Section 5 validate the above analysis and quantify the relative performance of SQ and VQ with and without receive filtering, for noisy DMCs.

### 4 Joint transmit–receive filtering and scaled codebook

In this section, we derive the joint transmit and receive linear filter and the scaled codebook that minimises the approximate expected distortion when the code index is transmitted over a noisy SEC. These approaches are feasible when the channel statistics $E_N$ is available at both the transmitter and receiver. Specifically, note that, in the system model introduced in Section 2, when $T = I$, the transmit filter drops out, and the system reduces to the receive-only filtering method considered in the previous section. When $R = I$ and $T = I$, where $0 \leq t \leq 1$ is a scale factor, the system reduces to the scaled codebook technique in [6]. When $R = T^{-1}R$, the system reduces to a transmit–receive filter with $T$ as the transmit filter and $R$ as the receive filter. For simplicity of exposition, we restrict to the MSE distortion and the i.i.d. isotropic standard Gaussian source. By the symmetry of the problem, it is sufficient to consider the transmit and receive filter to be scalar matrices, that is, $T = tI$ and $R = rI$.

From (1), the end-to-end MSE distortion is given by

$$
J = \text{tr}(S_{xx} - S_{xy}T - RS_{xy} + RS_{yy}R^T)
$$

(24)

It is shown in Appendix 3 that, under standard high-rate approximations

$$
S_{xy} = (1 - N E_N)(S_{xx} - \Theta_i)
$$

$$
S_{yy} = (1 - N E_N)(S_{xx} - \Theta_i) + N E_N S_{h_i}
$$

(25)

where $S_{h_i} \triangleq ((n + 2)/n)^2$ is the covariance of the transformed codebook. Also, $\Theta_i$ is defined as

$$
\Theta_i \triangleq \frac{2 m^2 N_l^{-2/3}}{n(1 - (2/(n + 2))^2)}
$$

(26)

with the constraint

$$
t \in \left( \sqrt{\frac{2}{n + 2}}, 1 \right]
$$

since the denominator in the above expression should remain non-negative for it to be meaningful.

Substituting the above expressions into (24), the approximate total distortion $E_d^{\text{LRF}}$ can be written as

$$
E_d^{\text{LRF}}(t) = n - (1 - N E_N)(2r - r^2)\text{tr}(I - \Theta_i) + N E_N r^2(n + 2)
$$

(27)

Taking the derivative of the above with respect to $r$ and equating to zero yields the receive filter $r_{\text{opt}}$ for a given value of $t$ as

$$
r_{\text{opt}} = \frac{(1 - N E_N)\text{tr}(I - \Theta_i)}{(1 - N E_N)\text{tr}(I - \Theta_i) + N E_N r_{\text{opt}}(n + 2)}
$$

(28)

Substituting the above $r_{\text{opt}}$ into (27) results in the approximate expected distortion with the receive filter optimised for a given $t$, as

$$
E_d^{\text{optRx}}(t) = n - \frac{(1 - N E_N)^2 (\text{tr}(I - \Theta_i))^2}{(1 - N E_N)\text{tr}(I - \Theta_i) + N E_N r_{\text{opt}}(n + 2)}
$$

(29)

Note that, (28) reduces to the LRF given by (4), and (29) reduces to the expression in (5), when $t = 1$ and $x$ is i.i.d. Gaussian. Now, the above expected distortion has to be numerically optimised over $t \in \left( \sqrt{2/(n + 2)}, 1 \right]$ to yield the jointly optimal transmit and receive filter and the corresponding minimum approximate MSE distortion.

Similarly, we can analyse the total distortion with the scaled codebook in [6], by simply setting $r = 1$ in (27), as follows

$$
E_d^{\text{CB}}(t) = n - (1 - N E_N)\text{tr}(I - \Theta_i) + N E_N r^2(n + 2)
$$

(30)
which can also be numerically optimised over $t \in \left(\sqrt{2/}n + 2\right)$ to find the scale factor that minimises the approximate expected distortion. Note that the above equation provides an analytical expression for optimising the scale factor, unlike the computationally cumbersome simulation-based approach adopted in [6, 7].

5 Simulation results

In this section, we validate the analytical expressions derived above and illustrate the improvement in the average distortion that can be obtained through the LRF, using Monte Carlo simulations. An $n$-dimensional i.i.d. zero mean Gaussian distributed vector with unit variance per dimension is used as the source and 50 000 instantiations are used for generating the optimal encoder codebook using the Lloyd-Max algorithm. The covariance of this source-optimised codebook is used as the covariance of the point density, for evaluating the theoretical expressions. Another set of 50 000 instantiations are used for encoding the source using the above computed codebook. The index from the encoder is sent over a noisy channel and the LRF is applied at the decoder before computing the end-to-end distortion. The noisy channel is modelled as a binary symmetric channel (BSC) with transition probability

$$q = Q(\sqrt{2SNR})$$

(23) that depends on the SNR per bit, and

$$B = \log_2 N$$

bits are employed for source quantisation.

Fig. 1 shows the average MSE and WMSE distortion for the source-optimised VQ with $n = 3$ and $B = 9$ bits. In the WMSE case, the matrix

$$W = \begin{bmatrix} 0.69 & 0.26 & 0.03 \\ 0.26 & 0.44 & 0.43 \\ 0.03 & 0.43 & 1.87 \end{bmatrix}$$

accurate to two decimals, was used. The matrix was generated using a random unitary matrix as eigenvectors and eigenvalues equal to $[2.0, 0.8, 0.2]$, which ensures that $\text{tr}(W)$ is the same as in the MSE case. The excellent match between the simulation results and the theoretical expression is clear from the figure. At low SNR, the distortion with and without filtering are found to approach $n$ and $2(n+1)$, which matches with (5) and (16), respectively. Also, at high SNR, all schemes converge to the high-rate distortion of VQ for noiseless channels, as expected.

Fig. 2 compares the MSE performance of SQ and VQ-based LRF for reducing the total distortion, with $n = 2$ and $B = 8$. It is interesting to note that the LRF greatly diminishes the performance difference between SQ and VQ. This corroborates with the theoretical expressions, since, at high rate, neglecting the terms of order $N^{-2/n}$, we have

$$E_{d, SQ}^{R_{opt}} - E_{d, VQ}^{R_{opt}} = \frac{2N\varepsilon_N(1-N\varepsilon_N)^2(n-1)}{(1+2N\varepsilon_N)(1+2N\varepsilon_N/n)}$$

Fig. 2 shows the performance comparison of the LRF designed for SQ and VQ with $n = 2$ and $B = 8$ bits.
followed by the performance of the joint transmit–receive linear filter, which also requires the feedback of the transmit filter. The joint transmit–receive linear filter outperforms the scaled codebook at all SNRs. The scaled codebook outperforms the LRF at high SNR, whereas the LRF performs better for SNR ≤ 4 dB. However, the scaled codebook requires numerical computation of the scale factor and its adaptation at both the transmitter and the receiver for every SNR. The LRF is the least computationally expensive and the simplest to implement. It always performs better than no filtering, and offers several dBs of improvement in the average MSE for practical SNR values.

Finally, Table 1 lists the percentage improvement in MSE distortion from the LRF compared to the no-filtering case. We observe that the percentage improvement is the highest for small N and n, and is higher for larger NEX. Moreover, in all three cases of NEX comparing the percentage improvement for N = 64 with N = 512, we observe that it has decreased for n = 1 and 2, but increased for n = 3. That is, as N is increased, the percentage improvement increases till about 4 bits per dimension, after which it starts to decrease. These observations agree with the theoretical expressions obtained above.

6 Conclusions

We considered the design and analysis of several adaptation techniques to mitigate the channel error induced distortion in FRQ with noisy SECs. We derived a closed-form expression for the linear receive-only filter that minimises the approximate expected WMSE distortion. We also derived analytical expressions for the average WMSE distortion of VQ and SQ, with and without the receive filter, using approximations from high-resolution quantisation theory. We extended the analysis to derive expressions for the MSE performance of VQ with joint transmit–receive filtering and with employing a scaled version of the codebook at both transmitter and receiver. We validated the analytical results using Monte Carlo simulations. We showed that a significant improvement in the MSE performance can be obtained at low SNR by using the LRF, when compared to the no filtering case. Future work could include the design of more sophisticated non-linear receive filtering methods, for example, using Occam filters [24].

Table 1 Percentage performance improvement because of receive filtering

<table>
<thead>
<tr>
<th>$\frac{\text{E}^{\text{opt}} - \text{E}^{\text{LRF}}}{\text{E}^{\text{LRF}}} \times 100$</th>
<th>$N_r N = 0.05$</th>
<th>$N_r N = 0.1$</th>
<th>$N_r N = 0.25$</th>
</tr>
</thead>
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<tr>
<td>Sim.</td>
<td>Th.</td>
<td>Sim.</td>
<td>Th.</td>
</tr>
<tr>
<td>n = 1, N = 64</td>
<td>6.65</td>
<td>6.72</td>
<td>13.63</td>
</tr>
<tr>
<td>n = 1, N = 512</td>
<td>2.77</td>
<td>2.82</td>
<td>5.72</td>
</tr>
<tr>
<td>n = 2, N = 64</td>
<td>3.97</td>
<td>4.13</td>
<td>9.12</td>
</tr>
<tr>
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<td>5.45</td>
</tr>
<tr>
<td>n = 3, N = 512</td>
<td>2.85</td>
<td>3.19</td>
<td>6.88</td>
</tr>
</tbody>
</table>

8 Appendices

8.1 Appendix 1. Review of high-rate distortion analysis

For a noiseless channel, the expected distortion of VQ can be obtained by taking the sum of the expected distortion in each of the N quantisation regions, as follows

\[
E_d = \sum_{i=1}^{N} \int_{R_i} \sum_{x \in R_i} d(x, \hat{x}) f_{x}(x) dx \tag{31}
\]

Now, for sufficiently large N, the specific point density (which defined as the piecewise constant function $g_N(x) = 1/(NV(R_i))$ for $x \in R_i$, where $V(R_i)$ is the
volume of the Voronoi region $\mathcal{R}_i$ is approximated by a continuous point density function $\lambda(x)$. Using this, (31) can be reduced to [1, 2, 25]

$$E_d^{SO} = \frac{n}{n + 2} N^{(-2/n)} \kappa_n^{(-2/n)} \int \lambda^{(-2/n)}(x) f(x) \, dx$$  \hspace{1cm} (32)

where ≈ denotes the asymptotic equality as $N$ obtains large, $\kappa_n$ is the volume of an $n$-dimensional unit sphere and the superscript on $E_d^{SO}$ stands for source-optimised VQ. The point density function that minimises (32) is given by the following expression [25]

$$\lambda^{opt}(x) = \frac{\lambda^{(n(n+2))}(x)}{\int_{D_n} \lambda^{(n(n+2))}(y) \, dy}$$  \hspace{1cm} (33)

The high-rate expected distortion for the special case of an $n$-dimensional i.i.d. standard Gaussian vector is given by

$$E_d^{SO} = 2 \pi N^{(-2/n)} \kappa_n^{(-2/n)} \left( \frac{n + 2}{n} \right)^{(n/2)} |W|^{(1/n)}$$  \hspace{1cm} (34)

### 8.2 Appendix 2. Some key approximations

We present some key high-rate approximations that are used in the derivation of the receive filter. Let $x \in \mathcal{R}_i$ be the source instantiation and let $\hat{x}_i$ be the received codeword when the index $i$ is transmitted over a noisy channel. Let $e = (x - \hat{x})$ denote the error vector. Then, the mean and covariance of $e$ can be written as

$$\mathbb{E}[e] = \sum_{i=1}^{N} \int_{\mathcal{R}_i} (x - \hat{x}) f(x) \, dx \approx 0$$  \hspace{1cm} (35)

$$\mathbb{E}[ee^T] = \sum_{i=1}^{N} \int_{\mathcal{R}_i} (x - \hat{x}) (x - \hat{x})^T f(x) \, dx \approx \sum_{i=1}^{N} \int_{\mathcal{E}_i} ee^T \, de$$  \hspace{1cm} (36)

where $\mathcal{E}_i$ denotes the Voronoi region $\mathcal{R}_i$ shifted to the origin. The equality (a) is obtained by assuming that the codewords are at the centroids of the Voronoi regions [2], and (36) is because of the approximating $f(x)$ with $f_{\hat{x}_i}(\hat{x})$ inside the quantisation cell $\mathcal{R}_i$ [2]. Using an ellipsoid approximation for $\mathcal{R}_i$, it can be shown that [11, 25]

$$\int_{T_0(W,V)} ee^T \, de = \frac{V_j}{n + 2} \left( \frac{\kappa_n}{\kappa_n^2} \right)^{(1/n)} \text{tr}(W^{-1}Q)$$  \hspace{1cm} (37)

where $Q$ is any positive semi-definite matrix and $T(y, W, V)$ is the hyper-ellipsoid defined as

$$T(y, W, V) \triangleq \left\{ x \left| \left( \frac{\kappa_n^2}{\kappa_n^2 V_j^2 W} \right)^{(1/n)} (x - y)^T W (x - y) \leq 1 \right. \right\}$$  \hspace{1cm} (38)

Now, to compute the $(i, j)$th element of the matrix in (36), we set $Q = E_{ij}$ in (37), with $E_{ij}$ being the all-zero matrix except for a 1 as the $(i, j)$th element, as follows

$$\sum_{i=1}^{N} f_i(\hat{x}_i) \int_{\mathcal{E}_i} ee^T \, de \approx \sum_{i=1}^{N} f_i(\hat{x}_i) \frac{V_j}{n + 2} \left( \frac{\kappa_n}{\kappa_n^2} \right)^{(1/n)} W^{-1} \approx \Phi_n \Gamma_n \lambda^{(-2/n)}$$  \hspace{1cm} (39)

where

$$\Phi_n \triangleq \kappa_n^{(-2/n)} |W|^{(1/n)} \left( \frac{n + 2}{n} \right)^{(n/2)}$$

$$\Gamma_n \triangleq \left[ \int_{D_n} f(x)^{(n/(n+2))} (x) \, dx \right]^{(n+2/n)}$$

and

$$\Theta \triangleq \Phi_n \Gamma_n \lambda^{(-2/n)}$$

In the above, (39) is obtained by substituting for the point density from (33) and converting the summation into the corresponding integral. We note that the approximations in obtaining the above expressions are very well accepted in the source coding literature [1, 2]. Further, the trace of the expression above results in the well-known high-rate characterisation of the WMSE of VQ-based source coding for noiseless channels [11, 25].

### 8.3 Appendix 3. Derivation of the joint transmit–receive filter and the scaled codebook

In this section, we derive the joint transmit and receive linear filter and the optimal scaled codebook that minimises the approximate expected end-to-end distortion when the code index is transmitted over a noisy symmetric error channel.

When the transmit filter $T$ is applied on the codebook at the encoder, $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N\} \triangleq \{\tilde{T}\tilde{x}_1, \tilde{T}\tilde{x}_2, \ldots, \tilde{T}\tilde{x}_N\}$ are used as the codebook for VQ encoding. Here $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N\}$ is the source-optimised VQ codebook. At the receiver, the codeword corresponding to the received index $j, \tilde{T}\tilde{x}_j$, is passed through a filter $\tilde{R}_j$ and $\tilde{R}\tilde{T}\tilde{x}_j$, is output as the estimate of the source instantiation.

We denote by $y$ the code vector corresponding to the received index, prior to receive filtering. Then, we can write $y = x + n$, where, for $x \in \mathcal{R}_i$, the noise vector is distributed as

$$f_n(n) = \begin{cases} \tilde{x}_j - x, & \text{with probability } (1 - (N - 1)e_N) \\ \tilde{x}_j - x, & \text{with probability } (N - 1)e_N \end{cases}$$  \hspace{1cm} (40)

where $\tilde{R}_i$ denotes the $i$th Voronoi region according to the transformed codebook

$$\tilde{R}_i \triangleq \left\{ x : (\tilde{x}_i - x)^T W (\tilde{x}_i - x) \leq 1 \right\}$$  \hspace{1cm} (41)

From (1), it is easy to see that the end-to-end WMSE distortion can be written as (24). We now work out the matrices $S_{xy}, S_{yy}$

$$S_{xy} = \mathbb{E}\{x(x + n)^T\} = S_{xx} + \mathbb{E}\{xn^T\}$$  \hspace{1cm} (42)

$$\mathbb{E}\{xn^T\} = (1 - (N - 1)e_N) \mathbb{E}\{x(\tilde{x}_i - x)^T\} + (N - 1)e_N \mathbb{E}\{x(\tilde{y} - x)^T\}$$  \hspace{1cm} (43)
where \( \hat{x}_i = T \hat{x}_i, \hat{y} = T \hat{x}_j \), for some \( j \neq i \). Now, \( \mathbb{E}\left\{ x(\hat{x}_i - x)^T \right\} \)
can be written as
\[
\mathbb{E}\left\{ x(\hat{x}_i - x)^T \right\} \approx -\frac{N^{-2/n} \kappa_n^{-2/n} |W|^{1/n}}{n + 2} W^{-1} \int_{x \in X} \lambda_T^{-2/n}(x)f(x) \, dx
\]
where
\[
\lambda_T(x) \triangleq \frac{1}{|T|} \lambda(T^{-1}x)
\]
is the point density of the transformed codebook. When \( W = I \) and \( x \) is isotropically distributed, by the symmetry of the problem, we can let \( T \) and \( \tilde{R} \) be scalar matrices, that is, \( T = tI \) and \( \tilde{R} = rI \). Hence, \( \lambda_T(x) \) has 0 mean and variance \((n + 2)/n^2\) per dimension. For the i.i.d. standard Gaussian source, (44) reduces to
\[
\mathbb{E}\left\{ x(\hat{x}_i - x)^T \right\} \approx -\frac{N^{-2/n} \kappa_n^{-2/n} |W|^{1/n}}{n + 2} W^{-1} \int_{x \in X} \lambda_T^{-2/n}(x)f(x) \, dx
\]
where \( t \in (\sqrt{2/(n + 2)}, 1] \), since the denominator in the above expression should remain non-negative for it to be meaningful. Note that, when \( t = 1 \), that is, with no transmit filtering, \( \Theta_i \), corresponds to the \( \Theta \) defined in Theorem 1, as expected. Similarly, it can be shown that
\[
\mathbb{E}\left\{ x(\hat{y} - x)^T \right\} \approx -\frac{N}{N-1} S_{xx} + \frac{1}{N-1} \Theta_i
\]
Substituting (45) and (46) in (43), we obtain
\[
\mathbb{E}\left\{ xn^T \right\} = -N \epsilon_N S_{xx} - (1 - N \epsilon_N) \Theta_i
\]
\[
S_{xy} = S_{xx} + \mathbb{E}\left\{ xn^T \right\} = (1 - N \epsilon_N)(S_{xx} - \Theta_i)
\]
Now, consider \( S_{yy} \triangleq \mathbb{E}\left\{ \hat{y} n^T \right\} \):
\[
\mathbb{E}\left\{ \hat{y} n^T \right\} = \mathbb{E}\left\{ (x + n)n^T \right\}
\]
\[
= \mathbb{E}\left\{ xy^T \right\} + \mathbb{E}\left\{ nx^T \right\} + \mathbb{E}\left\{ nn^T \right\}
\]
Note \( \mathbb{E}\left\{ xy^T \right\} \) and \( \mathbb{E}\left\{ xn^T \right\} \) are as computed above. It can be easily shown that
\[
\mathbb{E}\left\{ nn^T \right\} = (1 - N \epsilon_N) \Theta_i + N \epsilon_N \left(S_{xx} + S_{\lambda} \right)
\]
where \( S_{\lambda} \triangleq ((n + 2)/n^2)I \) is the covariance of the transformed codebook. Substituting (47), (48) and (50) in (49), we obtain
\[
S_{yy} = (1 - N \epsilon_N)(S_{xx} - \Theta_i) + N \epsilon_N S_{\lambda}
\]