Pseudocodewords of Tanner Graphs

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Abstract

Lower bounds on the minimal pseudocodeword weight for different channels are presented; analogous to minimum-distance codewords, minimal-weight pseudocodewords largely dominate the performance of the iterative decoder. Examining the structure of pseudocodewords of Tanner graphs, pseudocodewords and graph properties that are potentially problematic with min-sum iterative decoding are identified. An upper bound on the minimum degree lift needed to realize a given irreducible lift-realizable pseudocodeword is presented in terms of its maximal component. It is further conjectured that all irreducible lift-realizable pseudocodewords have components upper bounded by a finite value $t$ that is dependent on the graph structure. Finally, different Tanner graph representations of individual codes, and the resulting pseudocodeword distribution and iterative decoding performance of each representation are examined. Examples of Tanner graphs giving rise to different types of pseudocodewords are also presented. The results obtained provide some insights in relating the structure of pseudocodewords to the Tanner graph and suggest ways of designing Tanner graphs with good minimal pseudocodeword weight.

Index Terms

Low density parity check codes, pseudocodewords, iterative decoding, min-sum iterative decoder.

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I. INTRODUCTION

Iterative decoders have gained widespread attention due to their remarkable performance in decoding LDPC codes. However, analyzing their performance on finite length LDPC constraint graphs has nevertheless remained a formidable task. Wiberg’s dissertation [1] was among the earliest works in characterizing iterative decoder convergence on finite-length LDPC constraint graphs or Tanner graphs. Both [1] and [2] examine the convergence behavior of the min-sum iterative decoder [3] on cycle codes, a special class of LDPC codes having only degree two variable nodes, and they provide some necessary and sufficient conditions for the decoder to converge. Analogous works in [4] and [5] explain the behavior of iterative decoders using the terminology of covering spaces or lifts of the base Tanner graph. The common underlying idea in all these works is the role of pseudocodewords, most of which can be interpreted as valid codeword configurations existing among all lifts of the base graph, in determining decoder convergence.

Pseudocodewords of a Tanner graph play an analogous role in determining convergence of an iterative decoder as codewords do for a maximum likelihood decoder. In this paper, we study the structure of pseudocodewords of a Tanner graph assuming min-sum iterative decoding as in [1], and classify different types of pseudocodewords that can arise depending on the graph structure. We present some sufficient conditions for a given pseudocodeword, based on its composition, to be problematic with min-sum iterative decoding. We also identify certain subgraphs of the Tanner graph that give rise to problematic (or, bad) pseudocodewords. Our work is primarily with pseudocodewords arising out of finite degree graph covers of the base Tanner graph as opposed to pseudocodewords arising on the min-sum iterative decoder’s computation tree [1], [6]. We note that not all pseudocodewords are bad for iterative decoding, and this prompts us to investigate the structure of good versus bad pseudocodewords with respect to the iterative decoder.

Since the notion of distance is essentially related to any decoding technique, the iterative decoder is characterized by the pseudodistance rather than the codeword-(Euclidean/Hamming)-distance. Distance reduces to weight with respect to the all-zero codeword. In the context of iterative decoding, the minimal weight pseudocodeword [4] is more fundamental than the minimal weight codeword. It has been observed that pseudocodewords are essentially stopping sets [4] in the case of the binary erasure channel (BEC), and hence
the minimal pseudocodeword weight $w_{\text{min}}$ is equal to the minimum stopping set size $s_{\text{min}}$ on the BEC channel. This prompts us to examine how $w_{\text{min}}$ and $s_{\text{min}}$ relate over other channels such as the binary symmetric channel (BSC) and the additive white Gaussian noise (AWGN) channel. We present lower bounds on $w_{\text{min}}$ and further, we bound the minimum weight of good and bad pseudocodewords separately.

The set of pseudocodewords of a Tanner graph can be approximated by the set of pseudocodewords arising as codewords in graph lifts of the base Tanner graph. We therefore examine the minimal degree lifts needed to realize irreducible pseudocodewords in this set, which we call the set of lift-realizable pseudocodewords. Any pseudocodeword is composed of a finite number of irreducible pseudocodewords. Thus, characterizing irreducible pseudocodewords is sufficient to describe the set of all pseudocodewords that can arise. Moreover, the weight of any pseudocodeword is lower bounded by the minimum weight of its constituent irreducible pseudocodewords, implying that the irreducible pseudocodewords are the ones that are more likely to cause the decoder to fail to converge. A bound on the minimal degree lift needed to realize a given pseudocodeword is given in terms of its maximal component. We show that all lift-realizable irreducible pseudocodewords whose supports are minimal stopping sets cannot have any component larger than some finite number $t$ which depends on the structure of the graph. It is also conjectured that any lift-realizable irreducible pseudocodeword has components upper bounded by a finite value $t$ that depends on the structure of the graph. Examples of graphs with known $t$-values are stated.

[5] characterizes the set of pseudocodewords in terms of a polytope. The polytope of [5] encompasses all pseudocodewords that can be realized on finite degree graph covers of the base Tanner graph, but does not include all pseudocodewords that can arise on the decoder’s computation tree [1], [6]. In this paper, we investigate the usefulness of the graph-covers-polytope definition of [5], with respect to the min-sum iterative decoder, in characterizing the set of pseudocodewords of a Tanner graph. In particular, we give examples of computation trees that have several pseudocodeword configurations that may be bad for iterative decoding whereas the corresponding polytopes of these graph do not contain these bad pseudocodewords. We note however that this does not mean the polytope definition of pseudocodewords is inaccurate; rather, it is exact for the case of linear programming decoding [7], but incomplete for min-sum iterative decoding.
We then examine different graph representations of individual codes to understand what structural properties in the Tanner graph are important for the design of LDPC constraint graphs. In particular, we examine different graph representations of the \([7, 4, 3]\) and the \([15, 11, 3]\) Hamming codes and investigate how the structure of pseudocodewords from each representation affects iterative decoder behavior. Considering these examples along with the good-performing finite-geometry LDPC codes of \([8]\), we observe that despite a very small girth, redundancy in the Tanner graph representation can improve the distribution of pseudocodewords in the graph and hence, iterative decoding performance.

Examples of an LDPC constraint graph having all pseudocodewords with weight at least \(d_{\text{min}}\), an LDPC constraint graph with both good and low-weight (strictly less than \(d_{\text{min}}\)) bad pseudocodewords, and an LDPC constraint graph with all bad non-codeword pseudocodewords, are discussed. The results presented in the paper are highlighted through these examples.

Definitions and the necessary terminology are introduced in Section 2. Lower bounds on the pseudocodeword weight are derived in Section 3. Section 4 analyzes the structure of pseudocodewords realizable in lifts of general Tanner graphs. Section 5 continues the analysis in Section 4 by providing a bound on the minimum degree lift needed to realize a given lift-realizable irreducible pseudocodeword of a Tanner graph. The graph-covers-polytope definition of \([5]\) is examined in Section 6 using the \([4, 1, 4]\) repetition code as example. In Section 7, the performance of iterative decoding on different Tanner graph representations of individual codes is examined. It is shown for two example codes that redundancy in representation improves iterative decoding performance. Section 8 presents some example codes to illustrate the different types of pseudocodewords that can arise depending on the graph structure. We also examine the performance of these different examples to illustrate the effect of the different pseudocodewords on iterative decoding. Section 9 summarizes the results and concludes the paper. For readability, the proofs have been moved to the appendix.

II. BACKGROUND

In this section we establish the necessary terminology and notation that will be used in this paper, including an overview of pseudocodeword interpretations, iterative decoding algorithms, and pseudocodeword weights. Let \(V\) denote a set of \(n\) variable nodes and let \(U\) denote a set of \(m\) constraint nodes and let \(G = (V, U; E)\) be
a bipartite graph comprising of $V$ variable nodes, $U$ constraint nodes, and edges $E \subseteq \{(v, u) | v \in V, u \in U\}$, and representing a binary LDPC code $C$ with minimum distance $d_{\text{min}}$.

A. Pseudocodewords

1) Computation Tree Interpretation: Wiberg originally formulated pseudocodewords in terms of the computation tree, as described in [1]. Let $C(G)$ be the computation tree, corresponding to the min-sum iterative decoder, of the base LDPC constraint graph $G$ [1]. The tree is formed by enumerating the Tanner graph from an arbitrary variable node, called the root of the tree, up through the desired number of layers corresponding to decoding iterations. A computation tree enumerated for $\ell$ iterations and having variable node $v_i$ acting as the root node of the tree is denoted by $C_i(G)_\ell$. The shape of the computation tree is dependent on the scheduling of message passing used by the iterative decoder on the Tanner graph $G$. In Figure 1, the computation tree $C_2(G)_2$ is shown. The computation tree aids in the analysis of iterative decoding by describing the message-passing algorithm using a cycle-free graph. Since iterative decoding is exact on cycle-free graphs, the computation tree is a valuable tool in the exact analysis of iterative decoding on finite-length LDPC codes with cycles.

A valid assignment of variable nodes in a computation tree is one where all constraint nodes are satisfied. A codeword $c$ corresponds to a valid assignment on the computation tree, where for each $i$, all $v_i$ nodes in the
computation tree are assigned the same value; a pseudocodeword \( p \), on the other hand, corresponds to a valid assignment on the computation tree, where for each \( i \), not all \( v_i \) in the computation tree need be assigned the same value. In other words, if an assignment on \( C(G) \) corresponds to a codeword, then the local codeword configuration at a check node \( c_i \) on \( C(G) \) is the same as at any other copy of \( c_i \) on \( C(G) \). However, a valid assignment on \( C(G) \) corresponds to a pseudocodeword, when the local codeword-configurations may differ at different copies of the checks \( c_i \) on \( C(G) \).

A local configuration at a check node \( c_i \) is obtained by taking the average of the local-codeword-configurations at all copies of \( c_i \) on the computation tree. If the local configurations at all the check nodes of the graph \( G \) are consistent, then they yield a pseudocodeword vector \( p \) that lies in the polytope of [5] (see equation 2), and [5] shows that such a pseudocodeword \( p \) is realizable as a codeword on a lift graph of \( G \). If the local configurations are not consistent among all the check nodes, then there is no well-defined vector for the valid configuration (or, pseudocodeword) on the computation tree, and this pseudocodeword is not realizable on a lift graph of \( G \).

Let \( S \) be a subset of \( \{1, 2, \ldots, n\} \) and let \( p_S \) denote the vector obtained by restricting the vector \( p \) to only those components indicated by \( S \). Further, let \( N(j) \) denote the set of variable node neighbors of the check node \( c_j \) in \( G \). We introduce the notion of a vector representation for a pseudocodeword in the following manner. Given a valid assignment on the computation tree, a pseudocodeword vector \( p \) may be defined as \( p \in [0, 1]^n \) such that, for all \( j \), \( p_{N(j)} \) is a local configuration (possibly scaled) at check \( c_j \), and if \( v_i \) is a variable node in \( N(j_1) \) and \( N(j_2) \), then \( p_i \) assumes the same value in \( p_{N(j_1)} \) and \( p_{N(j_2)} \), for all variable nodes and their check node neighbors. If no such vector exists for a valid assignment on the computation tree, then we say that the corresponding pseudocodeword configuration has no vector representation.

2) Graph Covers Definition: A degree \( \ell \) cover (or, lift) \( \hat{G} \) of \( G \) is defined in the following manner:

**Definition 2.1:** A finite degree \( \ell \) cover of \( G = (V, U; E) \) is a bipartite graph \( \hat{G} \) where for each vertex \( x_i \in V \cup U \), there is a cloud \( \hat{X}_i = \{\hat{x}_{i1}, \hat{x}_{i2}, \ldots, \hat{x}_{i\ell}\} \) of vertices in \( \hat{G} \), with \( \text{deg}(x_i) = \text{deg}(x_{i\ell}) \) for all \( 1 \leq j \leq \ell \), and for every \( (x_i, x_j) \in E \), there are \( \ell \) edges from \( \hat{X}_i \) to \( \hat{X}_j \) in \( \hat{G} \) connected in a 1–1 manner.

Figure 2 shows a base graph \( G \) and a degree four cover of \( G \).
Definition 2.2: Suppose that \( \hat{c} = (\hat{c}_1, 1, \ldots, \hat{c}_1, \ell, \hat{c}_2, 1, \ldots, \hat{c}_2, \ell, \ldots) \) is a codeword in the Tanner graph \( \hat{G} \) representing a degree \( \ell \) lift of \( G \). A pseudocodeword \( p \) of \( G \) is a vector \((p_1, p_2, \ldots, p_n)\) obtained by reducing a codeword \( \hat{c} \), of the code in the lift graph \( \hat{G} \), in the following way:

\[
\hat{c} = (\hat{c}_1, 1, \ldots, \hat{c}_1, \ell, \hat{c}_2, 1, \ldots, \hat{c}_2, \ell, \ldots) \rightarrow (\hat{c}_1 + \hat{c}_1 + \ldots + \hat{c}_1, \ell, \hat{c}_2 + \hat{c}_2 + \ldots + \hat{c}_2, \ell, \ldots) = (p_1, p_2, \ldots, p_n) = p,
\]

where \( p_i = (\hat{c}_i, 1 + \hat{c}_i, 2 + \ldots + \hat{c}_i, \ell) \).

Note that each component of the pseudocodeword is merely the number of 1-valued variable nodes in the corresponding variable cloud, and that any codeword \( c \) is trivially a pseudocodeword as \( c \) is a valid codeword configuration in a degree-one lift. Equivalently, pseudocodewords realizable in graph covers can be defined by taking as components the fraction of one-valued variable nodes in every cloud. This definition yields a pseudocodeword \( p = (p_1, p_2, \ldots, p_n) \) that is a vector of rational entries such that \( p \in [0, 1]^n \), and

\[
p_i = \frac{\hat{c}_i, 1 + \hat{c}_i, 2 + \ldots + \hat{c}_i, \ell}{\ell}.
\]

Although this set of lift-realizable pseudocodewords is a subset of those arising in the computation tree, we will restrict the analysis of pseudocodewords to this set as it is easier to handle and besides, contains a significant fraction of pseudocodewords of a Tanner graph.

Definition 2.3: A pseudocodeword that does not correspond to a codeword in the base Tanner graph is called a non-codeword pseudocodeword, or nc-pseudocodeword, for short.

Polytope Representation

The set of all pseudocodewords associated with a given Tanner graph \( G \) has an elegant geometric description [5], [6], [7]. In [5], Koetter and Vontobel characterize the set of pseudocodewords via the fundamental cone.
For each parity check $j$ of degree $\delta_j$, let $C_j$ denote the $(\delta_j, \delta_j - 1, 2)$ simple parity check code, and let $P_{\delta_j}$ be a $2^{\delta_j-1} \times \delta_j$ matrix with the rows being the codewords of $C_j$. The fundamental polytope at check $j$ of a Tanner graph $G$ is then defined as:

$$ PMS(C_j) = \{ \omega \in \mathbb{R}^{\delta_j} : \omega = xP_{\delta_j}, x \in \mathbb{R}^{2^{\delta_j-1}}, 0 \leq x_i \leq 1, \sum_i x_i = 1 \}, \quad (1) $$

and the fundamental polytope of $G$ is defined as:

$$ PMS(G) = \{ \omega \in \mathbb{R}^n : \omega_{N(j)} \in PMS(C_j), j = 1, \ldots, n-k \}, \quad (2) $$

We use the superscript $MS$ to refer to min-sum iterative decoding and the notation $\omega_{N(j)}$ to denote the vector $\omega$ restricted to the coordinates of the neighbors of check $c_j$. The fundamental polytope gives a compact characterization of all possible lift-realizable pseudocodewords of a given Tanner graph $G$. Removing multiplicities of vectors, the fundamental cone $F(G)$ associated with $G$ is obtained as:

$$ F(G) = \{ \mu \omega \in \mathbb{R}^n : \omega \in PMS(G), \mu \geq 0 \}. $$

Though the fundamental cone is the same for all channels, the worst case pseudocodewords depend on the channel [5].

**Definition 2.4:** A pseudocodeword $p$ of the Tanner graph $G$ is said to be lift-realizable if $p$ is obtained by reducing a valid codeword configuration on some lift (or, cover) graph $\hat{G}$ of $G$ as described in Definition 2.2.

In other words, a lift-realizable pseudocodeword $p$ corresponds to a point in the graph-covers polytope $PM_{MS}(G)$.

In [7], Feldman also uses a polytope to characterize the pseudocodewords in Linear Programming (LP) decoding and this polytope has striking similarities with the polytope of [5]. Let $E(C_j)$ denote the set of all configurations that satisfy the code $C_j$ (as defined above). Then the feasible set of the LP decoder is given by:

$$ PLP(G) = \{ c \in \mathbb{R}^n : x_j \in \mathbb{R}^{2^{\delta_j-1}}, \sum_{S \in E(C_j)} x_{j,S} = 1, c_i = \sum_{S \in E(C_j), i \in S} x_{j,S} \forall i \in N(j), \quad \text{if } S \in E(C_j), j \in \{1, \ldots, n-k\} \} $$
Remark 2.1: It can be shown that the polytopes of [5] and [7] are equivalent, i.e., $P^{MS}(G) = P^{LF}(G)$ [9].

3) Relation to Stopping Sets:

Definition 2.5: The support of a vector $x = (x_1, \ldots, x_n)$ is the set of indices $i$ where $x_i \neq 0$.

Definition 2.6: [10] A stopping set in $G$ is a subset $S$ of $V$ where for each $s \in S$, every neighbor of $s$ is connected to $S$ at least twice.

The size of a stopping set $S$ is equal to the number of elements in $S$. A stopping set is said to be minimal if there is no smaller sized stopping set contained within it. The smallest minimal stopping set is called a minimum stopping set, and its size is denoted by $s_{\text{min}}$. Note that a minimum stopping set is not necessarily unique. Figure 3 shows a stopping set in the graph. Observe that $\{v_4, v_7, v_8\}$ and $\{v_3, v_5, v_9\}$ are two minimum stopping sets of size $s_{\text{min}} = 3$, whereas $\{v_0, v_1, v_3, v_5\}$ is a minimal stopping set of size 4.

One useful observation is that that the support of a lift-realizable pseudocodeword forms a stopping set in $G$.

Lemma 2.1: The support of a lift-realizable pseudocodeword $p$ of $G$ is the incidence vector of a stopping set in $G$.

4) Irreducibility:

Definition 2.7: A pseudocodeword $p = (p_1, \ldots, p_n)$ is irreducible if it cannot be written as a sum of two or more codewords or pseudocodewords.

Note that irreducible pseudocodewords are called minimal pseudocodewords in [5] as they correspond to vertices of the polytope $P^{MS}(G)$. We will see in subsequent sections that the irreducible pseudocodewords,
as defined above, are the ones that can potentially cause the min-sum decoder to fail to converge.

**B. Iterative Decoding**

The feature that makes LDPC codes attractive is the existence of computationally simple decoding algorithms. These algorithms either converge iteratively to a sub-optimal solution that may or may not be the maximum likelihood solution, or do not converge at all. The most common of these algorithms are the min-sum and the sum-product algorithms [3], [11]. These two algorithms are graph-based message-passing algorithms applied on the LDPC constraint graph. More recently, linear programming (LP) decoding has been applied to decode LDPC codes. Although LP decoding is more complex, it has the advantage that when it decodes to a codeword, the codeword is guaranteed to be the maximum-likelihood codeword (see [7]).

A message-passing decoder exchanges messages along the edges of the code’s constraint graph. For binary LDPC codes, the variable nodes assume the values one or zero; hence, a message can be represented either as the probability vector \([p_0, p_1]\), where \(p_0\) is the probability that the variable node assumes a value of 0, and \(p_1\) is the probability that the variable node assumes a value of 1, or as a log-likelihood ratio (LLR) \(\log\left(\frac{p_0}{p_1}\right)\), in which case the domain of the message is the entire real line \(\mathbb{R}\).

The two decoding algorithms are best described by the following update rules [11]:

- At a variable node \(v\) (Figure 4), a message sent out along an edge \(e\) is the result of a function whose input parameters are the messages received at \(v\) on edges other than \(e\), inclusive of the messages received from any external nodes connected to \(v\) (such as the one shown as a blank square in Figure 4). (The
external node in Figure 4 is introduced in the existing bipartite constraint graph strictly for convenience; it distinguishes messages that are obtained from the channel from messages that are obtained from neighboring constraint nodes.) A message from the external node in this case corresponds to channel information associated with the variable node $v$.

- At a constraint node $s$ (Figure 5), a message sent to a variable node along an edge $e$ is the result of a local function at $s$ whose input parameters are the messages received at $s$ on edges other than $e$.

Let $\mu[x\rightarrow f](x)$ denote the message (a log-likelihood ratio (LLR)) sent from the variable node $x$ to the constraint node $f$, and let $\mu[f\rightarrow x](x)$ be the message sent from node $f$ to node $x$. Let $n(v)$ denote the set of neighbors of node $v$ (i.e., nodes connected to $v$ by an edge), and let $n(v)\{u\}$ denote the set of neighbors of $v$ excluding node $u$.

1) Min-Sum decoding: The update rules in min-sum (MS) decoding are described below. For notation, let $y$ be the channel observation corresponding to a variable node $x$ and for convenience, we also use $x$ as the random variable associated with the variable node $x$. Further, let $Y$ be the set of random variables corresponding to the neighboring variable nodes of the constraint node $f$, i.e., $N(f)$, and let $g(Y)$ be an indicator function that is either 0 or $\infty$ depending on whether the arguments in $Y$ satisfy the constraint equation imposed by the node $f$ or not. We use the symbol $\sim \{x\}$ to indicate the set of all variables in $Y$ other than $x$. The update rules in MS decoding are then as follows:

- At a variable node at $x$:

$$\mu[x\rightarrow f](x) = -\log \left( \frac{P(y|x=1)}{P(y|x=0)} \right) + \sum_{h\in n(x)\{f\}} \mu[h\rightarrow x](x),$$

- At a constraint node $f$:

$$\mu[f\rightarrow x](x) = \min_{\sim \{x\}} (g(Y) \sum_{z\in n(f)\{x\}} \mu[z\rightarrow f](z)).$$

- Final decision at a variable node: After each iteration, the estimate at a variable node $x$ is

$$\hat{x} = \begin{cases} 
0 & \text{if } -\log \left( \frac{P(y|x=1)}{P(y|x=0)} \right) + \sum_{h\in n(x)} \mu[h\rightarrow x](x) \geq 0 \\
1 & \text{otherwise}
\end{cases}$$
Let $c = (c_1, \ldots, c_n)$ be a codeword and let $w = (w_1, \ldots, w_n)$ be the input to the decoder from the channel. That is, the LLR’s from the channel for the codebits $v_1, \ldots, v_n$ are $w_1, \ldots, w_n$, respectively. Then the optimal min-sum decoder (or, ML decoder) estimates the codeword

$$c^* = \arg \min_{c \in C} (c_1 w_1 + c_2 w_2 + \ldots + c_n w_n) = \arg \min_{c \in C} cw^T.$$ 

Let $P$ be the set of all pseudocodewords (including all codewords) of the graph $G$. Then the graph-based min-sum decoder described above estimates [5]

$$x^* = \arg \min_{x \in P} xw^T.$$ 

We will refer to the dot product $xw^T$ as the cost-function of the vector $x$ with respect to the channel input vector $w$. The optimal MS decoder estimates the codeword with the lowest cost whereas the sub-optimal graph-based iterative MS decoder tries to estimate the pseudocodeword with the lowest cost.

2) Sum-Product decoding: The update rules in sum-product (SP) decoding are as follows:

- At a variable node:
  $$\mu_{x \rightarrow f}(x) = -\log \left( \frac{P(y|x = 1)}{P(y|x = 0)} \right) + \sum_{h \in n(x) \setminus \{f\}} \mu_{h \rightarrow x}(x),$$

- At a constraint node:
  $$\mu_{f \rightarrow x}(x) = \sum_{z \sim \{x\}} g(Y) \prod_{z \in n(f) \setminus \{x\}} \mu_{z \rightarrow f}(z).$$

At a constraint node, the SP decoder computes the aposteriori probability of an adjoining variable node based on the estimates received from the remaining variable nodes that are connected to the constraint node. The SP decoder is sub-optimal in the sense that this computation is performed at every constraint independent of the other constraints. However, this leads to the optimal solution in the special case where the constraint graph has no closed paths (or, cycles).

The optimal SP decoder estimates the aposteriori probability of every bit node given the received LLR’s from the channel. The sub-optimal graph-based SP decoder, described above, also takes into account all pseudocodewords of the given graph in its estimate. Thus, its estimate may not always correspond to the estimate of the optimal decoder.
Note that min-sum versus sum-product decoding of a graph based code is analogous to Viterbi versus BCJR decoding of a trellis code.

C. Pseudocodewords and Decoding Behavior

For the rest of the paper, we will focus attention on the graph-based min-sum (MS) iterative decoder, since it is easier to analyze than the sum-product (SP) decoder. The following definition characterizes the iterative decoder behavior, providing conditions when the MS decoder may fail to converge to a valid codeword.

Definition 2.8: [2] A pseudocodeword $p = (p_1, p_2, \ldots, p_n)$ is good if for all input weight vectors $w = (w_1, w_2, \ldots, w_n)$ to the min-sum iterative decoder, there is a codeword $c$ that has lower overall cost than $p$, i.e., $cw^T < pw^T$.

Suppose the all-zeros codeword is the maximum-likelihood (ML) codeword for an input weight vector $w$, then all non-zero codewords $c$ have a positive cost, i.e., $cw^T > 0$.

Definition 2.9: A pseudocodeword $p$ is bad if there is a weight vector $w$ such that for all non-zero codewords $c$, $cw^T > pw^T$.

Again, in the case where the all-zeros codeword is the maximum-likelihood (ML) codeword, it is equivalent to say that a pseudocodeword $p$ is bad if there is a weight vector $w$ such that for all codewords $c$, $cw^T \geq 0$ but $pw^T < 0$.

Using the terminology of deviation sets, Wiberg [1, Theorem 4.2] states this necessary condition for the min-sum iterative-decoder to fail to converge, which can be rephrased as follows:

Theorem 2.1: A necessary condition for a decoding error to occur is that the cost of some irreducible pseudocodeword $p$ which is not a codeword is non-positive.

As in classical coding where the distance between codewords affects error correction capabilities, the distance between pseudocodewords affects iterative decoding capabilities. Analogous to the classical case, the distance between a pseudocodeword and the all-zeros codeword is captured by weight. The weight of a pseudocodeword depends on the channel, as noted in the following definition.

Definition 2.10: [4] Let $p = (p_1, p_2, \ldots, p_n)$ be a pseudocodeword of the code $C$ represented by the Tanner graph $G$, and let $e$ be the smallest number such that the sum of the $e$ largest $p_i$’s is at least $\sum_{i=1}^{n} p_i \geq \frac{\sum_{i=1}^{n} p_i}{2}$. Then the
the weight of $p$ is:

- $w_{\text{BEC}}(p) = |\text{supp}(p)|$ for the binary erasure channel (BEC);
- $w_{\text{BSC}}(p)$ for the binary symmetric channel (BSC) is:
  
  $w_{\text{BSC}}(p) = \begin{cases} 
  2e, & \text{if } \sum_{i} p_i = \frac{\sum_{i} p_i}{2} \\
  2e - 1, & \text{if } \sum_{i} p_i > \frac{\sum_{i} p_i}{2} 
  \end{cases}$

  where $\sum_{i} p_i$ is the sum of the $e$ largest $p_i$’s.

- $w_{\text{AWGN}}(p) = \frac{(p_1 + p_2 + \ldots + p_n)^2}{(p_1^2 + p_2^2 + \ldots + p_n^2)}$ for the additive white Gaussian noise (AWGN) channel.

Note that the weight of a pseudocodeword of $G$ reduces to the traditional Hamming weight when the pseudocodeword is a codeword of $G$, and that the weight is invariant under scaling of a pseudocodeword. The minimal pseudocodeword weight of $G$ is the minimum weight over all pseudocodewords of $G$ and is denoted by $w_{\text{BEC}}^{\text{min}}$ for the BEC channel (and likewise, for other channels). The minimal pseudocodeword weight $w_{\text{min}}$ is of fundamental importance as it plays an analogous role in iterative decoding as the minimum distance $d_{\text{min}}$ in ML-decoding.

Remark 2.2: The definition of pseudocodeword and pseudocodeword weights are the same for generalized Tanner graphs, wherein the constraint nodes represent subcodes instead of simple parity-check nodes. The difference is that as the constraints impose more conditions to be satisfied, there are fewer possible non-codeword pseudocodewords. Therefore, using stronger sub-codes in an LDPC constraint graph can only increase the minimum pseudocodeword weight.

On the erasure channel, pseudocodewords of $G$ are essentially stopping sets in $G$ [4], [7] and thus, the non-convergence of the iterative decoder is attributed to the presence of stopping sets. Moreover, any stopping set is potentially a bad pseudocodeword for the erasure channel since any stopping set can potentially prevent the iterative decoder from converging. In subsequent sections, we will examine the MS decoder on the BSC and the AWGN channels and the structure of pseudocodewords over these channels.

III. BOUNDS ON MINIMAL PSEUDOCODEWORD WEIGHTS

In this section, we derive lower bounds on the pseudocodeword weight for the BSC and AWGN channels, following Definition 2.10. The support size of a pseudocodeword $p$ has been shown to upper bound its weight...
on the BSC/AWGN channel [4]. Hence, from Lemma 2.1, it follows that $w_{\text{min}}^{\text{BSC/AWGN}} \leq s_{\text{min}}$. We establish the following lower bounds for the minimal pseudocodeword weight:

**Theorem 3.1:** Let $G$ be a regular bipartite graph with girth $g$ and smallest left degree $d$. Then the minimal pseudocodeword weight is lower bounded by

$$w_{\text{min}}^{\text{BSC/AWGN}} \geq \begin{cases} 1 + d + d(d - 1) + \ldots + d(d - 1)^{\frac{g-5}{4}}, & \frac{g}{2} \text{ odd} \\ 1 + d + \ldots + d(d - 1)\frac{g+4}{4} + (d - 1)\frac{g-4}{4}, & \frac{g}{2} \text{ even} \end{cases}.$$  

Note that this lower bound holds analogously for the minimum distance $d_{\text{min}}$ of $G$ [12], and also for the size of the smallest stopping set, $s_{\text{min}}$, in a graph with girth $g$ and smallest left degree $d$ [13].

For generalized LDPC codes, wherein the right nodes in $G$ of degree $k$ represent constraints of a $[k, k', \epsilon k]$ sub-code$^1$, the above result is extended as:

**Theorem 3.2:** Let $G$ be a $k$-right-regular bipartite graph with girth $g$ and smallest left degree $d$ and let the right nodes represent constraints of a $[k, k', \epsilon k]$ subcode, and let $x = (\epsilon k - 1)$. Then:

$$w_{\text{min}}^{\text{BSC/AWGN}} \geq \begin{cases} 1 + dx + d(d - 1)x^2 + \ldots + d(d - 1)^{\frac{g+4}{4}}x^\frac{g-4}{4}, & \frac{g}{2} \text{ odd} \\ 1 + dx + \ldots + d(d - 1)^{\frac{g-8}{4}}x^{\frac{g-8}{4}} + (d - 1)^{\frac{g-4}{4}}x^{\frac{g-4}{4}}, & \frac{g}{2} \text{ even} \end{cases}.$$  

In the generalized case, a stopping set may be defined as a set of variable nodes $S$ whose neighbors are each connected at least $\epsilon k$ times to $S$ in $G$. This definition makes sense since an optimal decoder on an erasure channel can recover at most $\epsilon k - 1$ erasures in a linear code of length $k$ and minimum distance $\epsilon k$. Thus if all constraint nodes are connected to a set $S$, of variable nodes, at least $\epsilon k$ times, and if all the bits in $S$ are erased, then the iterative decoder will not be able to recover any erasure bit in $S$. By the above definition of a stopping set in a generalized Tanner graph, a similar lower bound holds for $s_{\text{min}}$ also.

**Lemma 3.1:** The minimum stopping set size $s_{\text{min}}$ in a $k$-right-regular bipartite graph $G$ with girth $g$ and smallest left degree $d$, wherein the right nodes represent constraints of a $[k, k', \epsilon k]$ subcode, is lower bounded as:

$^1$Note that $\epsilon k$ and $\epsilon$ are the minimum distance and the relative minimum distance, respectively of the sub-code.
\[ s_{\min} \geq \begin{cases} 
1 + dx + d(d - 1)x^2 + \ldots + d(d - 1)^{\frac{g-6}{4}}x^{\frac{g-2}{4}}, & \frac{g}{2} \text{ odd} \\
1 + dx + \ldots + d(d - 1)^{\frac{g-8}{4}}x^{\frac{g-4}{4}} + (d - 1)^{\frac{g-4}{4}}x^\frac{g}{4}, & \frac{g}{2} \text{ even} 
\end{cases} \]

where \( x = (ek - 1) \).

The **max-fractional weight** of a vector \( x = [x_1, \ldots, x_n] \) is defined as \( w_{\max-\text{frac}}(x) = \frac{\sum_{i=1}^{n} x_i}{\max_{i} x_i} \). The max-fractional weight of pseudocodewords in LP decoding (see [7]) is analogous to the pseudocodeword weight in MS decoding.

It is worth noting that for any pseudocodeword \( p \), the pseudocodeword weight of \( p \) on the BSC and AWGN channels relates to the max-fractional weight of \( p \) as follows:

**Lemma 3.2:** For any pseudocodeword \( p \), \( w^{\text{BSC/AWGN}}(p) \geq w_{\max-\text{frac}}(p) \).

It follows that \( w_{\min}^{\text{BSC/AWGN}} \geq d_{\text{frac}}^{\max} \), the max-fractional distance which is the minimum max-fractional weight over all \( p \). Consequently, the bounds established in [7] for \( d_{\text{frac}}^{\max} \) are also lower bounds for \( w_{\min} \). One such bound is given by the following theorem.

**Theorem 3.3:** (Feldman [7]) Let \( \text{deg}_{l}^- \) (respectively, \( \text{deg}_{r}^- \)) denote the smallest left degree (respectively, right degree) in a bipartite graph \( G \). Let \( G \) be a factor graph with \( \text{deg}_{l}^- \geq 3, \text{deg}_{r}^- \geq 2 \), and girth \( g \), with \( g > 4 \). Then

\[
d_{\text{frac}}^{\max} \geq (\text{deg}_{l}^- - 1)^{\left\lceil \frac{g}{4} \right\rceil - 1}.
\]

**Corollary 3.4:** Let \( G \) be a factor graph with \( \text{deg}_{l}^- \geq 3, \text{deg}_{r}^- \geq 2 \), and girth \( g \), with \( g > 4 \). Then

\[
w_{\min}^{\text{BSC/AWGN}} \geq (\text{deg}_{l}^- - 1)^{\left\lceil \frac{g}{4} \right\rceil - 1}.
\]

Note that Corollary 3.4, which is essentially the result obtained in Theorem 3.1, makes sense due to the equivalence between the LP polytope and MS polytope (see Section 2).

We now bound the weight of a pseudocodeword \( p \) based on its maximal component value \( t \) and its support size \( |\text{supp}(p)| \). In Section 5, we conjecture that all irreducible pseudocodewords realizable in a finite cover cannot have any component larger than some fixed value \( t \); this value depends on the structure of the graph. Thus, when \( t \) is known, we can bound the weight of all irreducible pseudocodewords as follows:
Lemma 3.3: Suppose in an LDPC constraint graph $G$ every irreducible lift-realizable pseudocodeword $p = (p_1, p_2, \ldots, p_n)$ with support set $V$ has components $0 \leq p_i \leq t$, for $1 \leq i \leq n$, then:

(a) $w_{\text{AWGN}}(p) \geq \frac{2t^2}{(1+t^2)(t-1)+2t|V|}$, and

(b) $w_{\text{BSC}}(p) \geq \frac{1}{t}|V|$.

For many graphs, the $t$-value may be small and this makes the above lower bound large. Since the support of any pseudocodeword is a stopping set (Lemma 2.1), $w_{\text{min}}$ can be lower bounded in terms of $s_{\text{min}}$ and $t$. Thus, stopping sets are also important in the BSC and the AWGN channels.

Further, we can bound the weight of good and bad pseudocodewords (see Definitions 2.8, 2.9) separately, as shown below:

Theorem 3.5: For an $[n, k, d_{\text{min}}]$ code represented by an LDPC constraint graph $G$: (a) if $p$ is a good pseudocodeword of $G$, then $w_{\text{BSC/AWGN}}(p) \geq w_{\text{max}} - \frac{1}{t}p \geq d_{\text{min}}$, and (b) if $p$ is a bad pseudocodeword of $G$, then $w_{\text{BSC/AWGN}}(p) \geq w_{\text{max}} - \frac{1}{t}p \geq s_{\text{min}}$, where $t$ is as in the previous lemma.

Intuitively, it makes sense for good pseudocodewords, i.e., those pseudocodewords that are not problematic for iterative decoding, to have a weight larger than the minimum distance of the code, $d_{\text{min}}$. However, we note that bad pseudocodewords can also have weight larger than $d_{\text{min}}$.

IV. STRUCTURE OF PSEUDOCODEWORDS

This section examines the structure of lift-realizable pseudocodewords and identifies some sufficient conditions for certain pseudocodeword to potentially cause the min-sum iterative decoder to fail to converge to a codeword. Some of these conditions relate to subgraphs of the base Tanner graph. As in the previous section, let $G$ be a bipartite graph representing a binary LDPC code $C$, with $|V| = n$ left (variable) nodes, $|U| = m$ right (check) nodes, and edges $E \subseteq \{(v, u) | v \in V, u \in U\}$. We recall that we are only considering the set of pseudocodewords that arise from finite degree lifts of the base graph, and that by Definition 2.2, the pseudocodewords have non-negative integer components.

Lemma 4.1: Let $p = (p_1, p_2, \ldots, p_n)$ be a pseudocodeword in the graph $G$ that represents the LDPC code $C$. Then the vector $x = p \mod 2$, obtained by reducing the entries in $p$, modulo 2, corresponds to a codeword in $C$.

The following implications follow from the above lemma:
• If a pseudocodeword \( p \) has at least one odd component, then it has at least \( d_{\text{min}} \) odd components.

• If a pseudocodeword \( p \) has a support size \( |\text{supp}(p)| < d_{\text{min}} \), then it has no odd components.

• If a pseudocodeword \( p \) has no non-zero codeword contained in its support, then it has no odd components.

**Lemma 4.2:** A pseudocodeword \( p = (p_1, \ldots, p_n) \) can be written as \( p = c^{(1)} + c^{(2)} + \ldots + c^{(k)} + r \), where \( c^{(1)}, \ldots, c^{(k)} \), are \( k \) (not necessarily distinct) codewords and \( r \) is some residual vector, containing no codeword in its support, that remains after removing the codewords \( c^{(1)}, \ldots, c^{(k)} \) from \( p \). Either \( r \) is the all-zero vector, or \( r \) is a vector comprising of 0 or even entries only.

This lemma describes a particular composition of a pseudocodeword \( p \). Note that the above result does not claim that \( p \) is reducible even though the vector \( p \) can be written as a sum of codeword vectors \( c^{(1)}, \ldots, c^{(k)} \), and \( r \). Since \( r \) need not be a pseudocodeword, it is not necessary that \( p \) be reducible *structurally* as a sum of codewords and/or pseudocodewords (as in Definition 2.7).

It is also worth noting that the decomposition of a pseudocodeword, even that of an irreducible pseudocodeword, is not unique.

**Example 4.1:** For representation B of the \([7, 4, 3]\) Hamming code as shown in Figure 13, label the vertices clockwise from the top as \( v_1, v_2, v_3, v_4, v_5, v_6, \) and \( v_7 \). The vector \( p = (p_1, \ldots, p_n) = (1, 2, 1, 1, 0, 2) \) is an irreducible pseudocodeword and may be decomposed as \( p = (1, 0, 1, 0, 0, 0, 1) + (0, 0, 0, 1, 1, 0, 1) + (0, 2, 0, 0, 0, 0, 0) \) and also as \( p = (1, 0, 1, 1, 0, 0) + (0, 2, 0, 0, 0, 0, 2) \). In each of these decompositions, each vector in the sum is a codeword except for the last vector which is the residual vector \( r \).

**Theorem 4.1:** Let \( p = (p_1, \ldots, p_n) \) be a pseudocodeword. If there is a decomposition of \( p \) as in Lemma 4.2 such that \( r = 0 \), then \( p \) is a good pseudocodeword as in Definition 2.8.

**Theorem 4.2:** The following are sufficient conditions for a pseudocodeword \( p = (p_1, \ldots, p_n) \) to be bad, as in Definition 2.9:

1) \( w^{BSC/AWGN}(p) < d_{\text{min}} \).

2) \( |\text{supp}(p)| < d_{\text{min}} \).

3) If \( p \) is a non-codeword irreducible pseudocodeword and \( |\text{supp}(p)| \geq \ell + 1 \), where \( \ell \) is the number of distinct codewords whose support is contained in \( \text{supp}(p) \).
Subgraphs of the LDPC constraint graph may also give rise to bad pseudocodewords, as indicated below.

Definition 4.1: A stopping set \( S \) has property \( \Theta \) if \( S \) contains at least one pair of variable nodes \( u \) and \( v \) that are not connected by any path that traverses only via degree two check nodes in the subgraph \( G_{|S} \) of \( G \) induced by \( S \) in \( G \).

Example 4.2: In Figure 26, the set \( \{v_1, v_2, v_4\} \) is a minimal stopping set and does not have property \( \Theta \), whereas the set \( \{v_1, v_3, v_4, v_5, v_6, v_7, v_{10}, v_{11}, v_{12}, v_{13}\} \) is not minimal but has property \( \Theta \). The graph in Figure 6 is a minimal stopping set that has property \( \Theta \).

![Fig. 6. A minimal stopping set with property \( \Theta \).](image)

Definition 4.2: A variable node \( v \) in an LDPC constraint graph \( G \) is said to be problematic if there is a stopping set \( S \) containing \( v \) that is not minimal but nevertheless has no proper stopping set \( S' \subset S \) for which \( v \in S' \).

Examples of graphs containing problematic nodes are presented in Section 8. Note that if a graph \( G \) has a problematic node, then \( G \) necessarily contains a stopping set with property \( \Theta \).

The following result classifies bad non-codeword pseudocodewords, with respect to the AWGN channel, using the graph structure of the underlying pseudocodeword supports, which, by Lemma 2.1, are stopping sets in the LDPC constraint graph.

Theorem 4.3: Let \( G \) be an LDPC constraint graph representing an LDPC code \( C \), and let \( S \) be a stopping set in \( G \). Then, the following hold:

1) If there is no non-zero codeword in \( C \) whose support is contained in \( S \), then all non-codeword pseudocodewords of \( G \), having support equal to \( S \), are bad as in Definition 2.9. Moreover, there exists a bad pseudocodeword in \( G \) with support equal to \( S \).

2) If there is at least one codeword \( c \) whose support is contained in \( S \), then we have the following cases:
   
   (a) if \( S \) is minimal,
(i) there exists a non-codeword pseudocodeword $p$ with support equal to $S$ iff $S$ has property $\Theta$.
(ii) all non-codeword pseudocodewords with support equal to $S$ are bad.

(b) if $S$ is not minimal,

(i) and $S$ contains a problematic node $v$ such that $v \notin S'$ for any proper stopping set $S' \subset S$, then there exists a bad pseudocodeword $p$ with support $S$. Moreover, any non-codeword irreducible pseudocodeword $p$ with support $S$ is bad.
(ii) and $S$ does not contain any problematic nodes, then every variable node in $S$ is contained in a minimal stopping set within $S$. Moreover, there exists a bad non-codeword pseudocodeword with support $S$ iff either one of these minimal stopping sets is not the support of any non-zero codeword in $C$ or one of these minimal stopping sets has property $\Theta$.

The graph in Figure 7 is an example of case 2(b)(ii) in Theorem 4.3. Note that the stopping set in the figure is a disjoint union of two codeword supports and therefore, there are no non-codeword irreducible pseudocodewords.

![Fig. 7. A non-minimal stopping set as in case 2(b)(ii) of Theorem 4.3.](image)

The graph in Figure 8 is an example of case 2(a). The graph has property $\Theta$ and therefore has non-codeword pseudocodewords, all of which are bad.

![Fig. 8. A minimal stopping set as in case 2(a) of Theorem 4.3.](image)
A. Remarks on the weight vector and channels

In [6], Frey et al. show that the max-product iterative decoder (equivalently, the min-sum iterative decoder) will always converge to an irreducible pseudocodeword on the AWGN channel. However, their result does not explicitly show that for a given irreducible pseudocodeword \( p \), there is a weight vector \( w \) such that the cost \( pw^T \) is the smallest among all possible pseudocodewords. In the previous subsection, we have given sufficient conditions under which such a weight vector can explicitly be found for certain irreducible pseudocodewords. We believe, however, that finding such a weight vector \( w \) for any irreducible pseudocodeword \( p \) may not always be possible.

In particular, we state the following definitions and results.

Definition 4.3: A truncated AWGN channel, parameterized by \( L \) and denoted by \( TAWGN(L) \), is an AWGN channel whose output log-likelihood ratios corresponding to the received values from the channel are truncated, or limited, to the interval \([-L, L]\).

In light of [14], [15], we believe that there are fewer problematic pseudocodewords on the BSC channel than on the truncated AWGN channel or the AWGN channel.

Definition 4.4: For an LDPC constraint graph \( G \) that defines an LDPC code \( C \), let \( P^B_{AWGN}(G) \) be the set of lift-realizable pseudocodewords of \( G \) where for each pseudocodeword \( p \) in the set, there exists a weight vector \( w \) such that the cost \( pw^T \) on the AWGN channel is the smallest among all possible lift-realizable pseudocodewords in \( G \).

Let \( P^B_{BSC}(G) \) and \( P^B_{TAWGN(L)}(G) \) be defined analogously for the BSC and the truncated AWGN channels, respectively. Then, we have the following result:

Theorem 4.4: For an LDPC constraint graph \( G \), and \( L \geq 1 \), we have

\[
P^B_{BSC}(G) \subseteq P^B_{TAWGN(L)}(G) \subseteq P^B_{AWGN}(G).
\]

The above result implies that there may be fewer problematic irreducible pseudocodewords for the BSC channel than over the TAWGN(L) channel and the AWGN channel. In other words, min-sum iterative decoding may be more accurate for the BSC channel than over the AWGN channel. Thus, quantizing or truncating the
received information from the channel to a smaller interval before performing min-sum iterative decoding may be beneficial. Since the set of lift-realizable pseudocodewords for min-sum iterative decoding is the set of pseudocodewords for linear-programming (LP) decoding (see Section 2), the same analogy carries over to LP decoding as well. Indeed, at high enough signal to noise ratios, the above observation has been shown true for the case of LP decoding in [14] and more recently in [15].

V. PSEUDOCODEWORDS AND LIFT-DEGREES

It is of interest to relate lift-realizable pseudocodewords with the lift-degrees on which they may be realized.

We start with the following upper bound on the smallest lift degree, denoted by $m_{\min}$, needed to realize a given pseudocodeword $p = (p_1, \ldots, p_n)$.

**Theorem 5.1:** Let $p = (p_1, \ldots, p_n)$ be an irreducible lift-realizable pseudocodeword of a graph $G$ having largest right degree $d_i^+$, and let $0 \leq p_i \leq t$, for $i = 1, \ldots, n$. Then the smallest lift degree $m_{\min}$ needed to realize $p$ satisfies

$$m_{\min} \leq \max_{h_j} \frac{\sum_{v_i \in N(h_j)} p_i}{2} \leq \frac{td_i^+}{2},$$

where the maximum is over all check nodes $h_j$ in the graph and $N(h_j)$ denotes the variable node neighbors of $h_j$.

**Remark 5.1:** If $p$ is any lift-realizable pseudocodeword and $b$ is the maximum component, then the smallest lift degree needed to realize $p$ is at most $\frac{bd_i^+}{2}$.

Recall that any pseudocodeword can be expressed as a sum of irreducible pseudocodewords, and further, that the weight of any pseudocodeword is lower bounded by the smallest weight of its constituent pseudocodewords. Therefore, given a graph $G$, it is useful to find the smallest lift degree needed to realize all irreducible lift-realizable pseudocodewords (and hence, also all minimum weight pseudocodewords).

One parameter of interest is the maximum component $t$ which can occur in any irreducible lift-realizable pseudocodeword of a given graph $G$, i.e., if a pseudocodeword $p$ has a component larger than $t$, then $p$ is reducible. If such a $t$ exists and is known, then Theorem 5.1 may be used to obtain the smallest lift degree needed to realize all irreducible lift-realizable pseudocodewords. This has great practical implications, for an
upper bound on the lift degree needed to obtain a pseudocodeword of minimal weight would significantly lower the complexity of determining $w_{\text{min}}$. Moreover, Theorem 3.5 shows how the weight of a pseudocodeword $p$ is lower bounded in terms of $t$ and the support size of $p$.

**Example 5.1:** Some graphs with known $t$-values are: $t = 2$ for cycle codes [1], [2], $t = 1$ for LDPC codes whose Tanner graphs are trees, and $t = 2$ for LDPC graphs having a single cycle.

For any finite bipartite graph, we have the following conjecture:

**Conjecture 5.1:** Every finite bipartite graph $G$ representing a finite length LDPC code has a finite $t$.

In [1], Wiberg shows that there are infinitely many irreducible pseudocodewords, or deviation sets, on the computation tree when the tree grows with increasing decoding iterations. Since the graph is finite, this means that some components must assume arbitrarily large values. However, in the graph covers setting, we believe that the structure of the graph ensures that at some large value, $t$, which may be exponentially large in the code length and the degree distribution of the graph, all the irreducible pseudocodewords are realizable. Some partial results are stated below, where we find bounds on the maximal component allowable in irreducible pseudocodewords with supports equal to stopping sets with special properties. If a graph $G$ contains only stopping sets of this form, then $t$ is finite. Unfortunately, the most likely case, that $S$ is not minimal, still remains open.

**Theorem 5.2:** Let $S$ be a stopping set. Let $t_S$ denote the largest component an irreducible pseudocodeword with support $S$ may have in the graph $G = S$. Then, the following hold:

1) If $S$ is a minimal stopping set and does not have property $\Theta$, then a pseudocodeword with support $S$ has maximal component 1 or 2. That is, $t_S = 1$ or 2.

2) If $S$ is minimal and has property $\Theta$, then $t_S$ is finite.

**VI. Graph-Covers-Polytope Approximation**

In this section, we examine the graph-covers-polytope definition of [5] in characterizing the set of pseudocodewords of a Tanner graph with respect to min-sum iterative decoding. Consider the $[4,1,4]$-repetition code which has a Tanner graph representation as shown in Figure 9. The corresponding computation tree for three iterations of message passing is also shown in the figure. The only lift-realizable pseudocodewords for
this graph are \((0, 0, 0, 0)\) and \((k, k, k, k)\), for some positive integer \(k\); thus, this graph has no non-codeword pseudocodewords. Even on the computation tree, the only valid assignment assigns the same value for all the nodes on the computation tree. Therefore, there are no non-codeword pseudocodewords on the graph’s computation tree as well.

![Tanner graph and computation tree (CT) for the \([4,1,4]\) repetition code.](image)

Suppose we add a redundant check node to the graph, then we obtain a new LDPC constraint graph, shown in Figure 10, for the same code. Even on this graph, the only lift realizable pseudocodewords are \((0, 0, 0, 0)\) and \((k, k, k, k)\), for some positive integer \(k\). Therefore the polytope of [5] contains \((0, 0, 0, 0)\) and \((1, 1, 1, 1)\) as the vertex points and has no bad pseudocodewords (as in Definition 2.9). However, on the computation tree, there are several valid assignments that do not have an equivalent representation in the graph-covers-polytope. The assignment where all nodes on the computation tree are assigned the same value, say 1, (as highlighted in Figure 10) corresponds to a codeword in the code. (For this assignment on the computation tree, the local configuration at check \(c_1\) is \((1,1)\) corresponding to \((v_1, v_2)\), at check \(c_2\), it is \((1, 1)\) corresponding to \((v_2, v_3)\), at check \(c_3\), it is \((1, 1)\) corresponding to \((v_3, v_4)\), at check \(c_4\), it is \((1, 1)\) corresponding to \((v_1, v_4)\), and at check \(c_5\), it is \((1, 1, 1, 1)\) corresponding to \((v_1, v_2, v_3, v_4)\). Thus, the pseudocodeword vector \((1, 1, 1, 1)\) corresponding to \((v_1, v_2, v_3, v_4)\) is consistent locally with all the local configurations at the individual check nodes.)

However, an assignment where some nodes are assigned different values compared to the rest (as highlighted in Figure 11) corresponds to a non-codeword pseudocodeword on the Tanner graph. (For the assignment shown...
in Figure 11, the local configuration at check $c_1$ is $(1, 1)$, corresponding to $(v_1, v_2)$, as there are two check nodes $c_1$ in the computation tree with $(1, 1)$ as the local codeword at each of them. Similarly, the local configuration at check $c_2$ is $(2/3, 2/3)$, corresponding to $(v_2, v_3)$, as there are three $c_2$ nodes on the computation tree, two of which have $(1, 1)$ as the local codeword and the third has $(0, 0)$ as the local codeword. Similarly, the local configuration at check $c_3$ is $(1/3, 1/3)$ corresponding to $(v_3, v_4)$, the local configuration at check $c_4$ is $(1/2, 1/2)$ corresponding to $(v_1, v_4)$, and the local configuration at check $c_5$ is $(1/3, 1, 0, 2/3)$ corresponding to $(v_1, v_2, v_3, v_4)$. Thus, there is no pseudocodeword vector that is consistent locally with all the above local configurations at the individual check nodes.)

Clearly, as the computation tree grows with the number of decoding iterations, the number of non-codeword pseudocodewords in the graph grows exponentially with the depth of the tree. Thus, even in the simple case of the repetition code, the graph-covers-polytope of [5] fails to capture all min-sum-iterative-decoding-pseudocodewords of a Tanner graph.

Fig. 10. Modified Tanner graph and CT for the [4,1,4] repetition code.

Fig. 11. Modified Tanner graph and CT for the [4,1,4] repetition code.
Figure 12 shows the performance of min-sum iterative decoding on the constraint graphs of Figures 9 and 10 when simulated over the binary input additive white Gaussian noise channel (BIAWGNC) with signal to noise ratio $E_b/N_0$. The maximum-likelihood (ML) performance of the code is also shown as reference. With a maximum of $10^4$ decoding iterations, the performance obtained by the iterative decoder on the single cycle constraint graph of Figure 9 is the same as the optimal ML performance (the two curves are one on top of the other), thereby, confirming that the graph has no nc-pseudocodewords. The iterative decoding performance deteriorates when a new degree four check node is introduced as in Figure 10. (A significant fraction of detected errors, i.e., errors due to the decoder not being able to converge to any valid codeword within $10^4$ iterations, were obtained upon simulation of this new graph.) To de-emphasize the effect of non-codeword pseudocodewords, arising out of the computation tree of Figure 10, on the iterative decoder, the message log-likelihood ratios were suitably scaled, as suggested in [6], and this pushed the performance to the optimal ML performance.
This example illustrates that the fundamental polytope of [5] does not capture the entire set of min-sum-iterative-decoding-pseudocodewords of a Tanner graph. In general, we state the following results:

Claim 6.1: A bipartite graph $G$ representing an LDPC code $C$ contains no non-codeword irreducible pseudocodewords on the computation tree $C(G)$ of any depth if and only if either (i) $G$ is a tree, or (ii) $G$ contains only degree two check nodes.

Claim 6.2: A bipartite graph $G$ representing an LDPC code $C$ contains no non-codeword irreducible lift-realizable pseudocodewords if either (i) $G$ is a tree, or (ii) there is at least one path between any two variable nodes in $G$ that traverses only via check nodes having degree two.

Note that condition (ii) in Claim 6.2 states that if there is at least one path between every pair of variable nodes that has only degree two check nodes, then $G$ contains no non-codeword irreducible lift-realizable pseudocodewords. However, condition (ii) in Claim 6.1 requires that every path between every pair of variable nodes has only degree two check nodes. Unlike Claim 6.1 which gives a necessary and sufficient condition, Claim 6.2 only provides a sufficient condition for a graph $G$ to contain no non-codeword irreducible pseudocodewords. Example 1 of Section 8 has no non-codeword irreducible lift-realizable pseudocodewords and yet does not meet the condition in Claim 6.2.

The above two claims state some conditions under which a graph has no non-codeword irreducible pseudocodewords. However, it would be interesting to find necessary and sufficient conditions for a graph to have no bad pseudocodewords with respect to the min-sum iterative decoder.

VII. GRAPH REPRESENTATIONS AND WEIGHT DISTRIBUTION

In this section, we examine different representations of individual LDPC codes and analyze the weight distribution of lift-realizable pseudocodewords in each representation and how it affects the performance of the min-sum iterative decoder. We use the classical $[7, 4, 3]$ and the $[15, 11, 3]$ Hamming codes as examples.

Figure 13 shows three different graph representations of the $[7, 4, 3]$ Hamming code. We will call the representations $A$, $B$, and $C$, and moreover, for convenience, also refer to the graphs in the three respective
Fig. 13. Three different representations of the [7,4,3] Hamming code.

representations as $A$, $B$, and $C$. The graph $A$ is based on the systematic parity check matrix representation of the [7, 4, 3] Hamming code and hence, contains three degree one variable nodes, whereas the graph $B$ has no degree one nodes and is more structured (it results in a circulant parity check matrix) and contains 4 redundant check equations compared to $A$, which has none, and $C$, which has one. In particular, $A$ and $C$ are subgraphs of $B$, with the same set of variable nodes. Thus, the set of lift-realizable pseudocodewords of $B$ is contained in the set of lift-realizable pseudocodewords of $A$ and $C$, individually. Hence, $B$ has fewer number of lift-realizable pseudocodewords than $A$ or $C$. In particular, we state the following result:

**Theorem 7.1:** The number of lift-realizable pseudocodewords in an LDPC graph $G$ can only reduce with the addition of redundant check nodes to $G$.

The proof is obvious since with the introduction of new check nodes in the graph, some previously valid pseudocodewords may not satisfy the new set of inequality constraints imposed by the new check nodes. (Recall that at a check node $c$ having variable node neighbors $v_{i_1}, \ldots, v_{i_k}$, a pseudocodeword $p = (p_1, \ldots, p_n)$, must satisfy the following inequalities $p_{i_j} \leq \sum_{h \neq j, h=1, \ldots, k} p_{i_h}$, for $j = 1, \ldots, k$ [5].) However, the set of valid codewords in the graph remains the same, since we are introducing only redundant (or, linearly dependent) check nodes. Thus, a graph with more check nodes can only have fewer number of lift-realizable pseudocodewords and possibly a better pseudocodeword-weight distribution.

If we add all possible redundant check nodes to the graph, which, we note, is an exponential number in the number of linearly dependent rows of the parity check matrix of the code, then the resulting graph would have the smallest number of lift-realizable pseudocodewords among all possible representations of the code. If this graph does not have any bad nc-pseudocodewords (both, lift-realizable ones and those arising on
the computation tree) then the performance obtained with iterative decoding is the same as the optimal ML performance.

**Remark 7.1:** Theorem 7.1 considers only the set of lift-realizable pseudocodewords of a Tanner graph. On adding redundant check nodes to a Tanner graph, the shape of the computation tree is altered and thus, it is possible that some new pseudocodewords arise in the altered computation tree, which can possibly have an adverse effect on iterative decoding. The \([4,1,4]\) repetition code example from Section 6 illustrates this. Iterative decoding is optimal on the single cycle representation of this code. However, on adding a degree four redundant check node, the iterative decoding performance deteriorates due to the introduction of bad pseudocodewords to the altered computation tree. (See Figure 12.) (The set of lift-realizable pseudocodewords however remains the same for the new graph with redundant check nodes as for the original graph.)

Returning to the Hamming code example, graph \(B\) can be obtained by adding edges to either \(A\) or \(C\), and thus, \(B\) has more number of cycles than \(A\) or \(C\). The distribution of the weights of the irreducible lift-realizable pseudocodewords for the three graphs \(A\), \(B\), and \(C\) is shown\(^2\) in Figure 14. (The distribution considers all irreducible pseudocodewords in the graph, since irreducible pseudocodewords may potentially prevent the min-sum decoder to converge to any valid codeword [6].) Although, all three graphs have a pseudocodeword of weight three\(^3\), Figure 14 shows that \(B\) has most of its lift-realizable pseudocodewords of high weight, whereas \(C\), and more particularly, \(A\), have more low-weight lift-realizable pseudocodewords. The corresponding weight distributions over the BEC and the BSC channels are shown in Figure 15. \(B\) has a better weight distribution than \(A\) and \(C\) over these channels as well.

The performance of min-sum iterative decoding of \(A\), \(B\), and \(C\) on the binary input (BPSK modulated) additive white Gaussian noise (AWGN) channel with signal to noise ratio (SNR) \(E_b/N_o\) is shown in Figures 16,

\(^2\)The plots considered all pseudocodewords in the three graphs that had a maximum component value of at most 3. The \(t\)-value (see Section 5) is 3 for the graphs \(A\), \(B\), and \(C\) of the \([7,4,3]\) Hamming code.

\(^3\)Note that this pseudocodeword is a valid codeword in the graph and is thus a *good* pseudocodeword for iterative decoding.
17, and 18, respectively. (The maximum number of decoding iterations was fixed at 100.) The performance plots show both the bit error rate and the frame error rate, and further, they also distinguish between undetected decoding errors, that are caused due to the decoder converging to an incorrect but valid codeword, and detected errors, that are caused due to the decoder failing to converge to any valid codeword within the maximum specified number of decoding iterations, 100 in this case. The detected errors can be attributed to the existence of non-codeword pseudocodewords and the decoder trying to converge to one of them rather than to any valid codeword.

Representation A has a significant detected error rate, whereas representation B shows no presence of detected errors at all. All errors in decoding B were due to the decoder converging to a wrong codeword. (We note that an optimal ML decoder would yield a performance closest to that of the iterative decoder on representation B.) This is interesting since the graph B is obtained by adding 4 redundant check nodes to the graph A. The addition of these 4 redundant check nodes to the graph removes most of the low-weight nc-pseudocodewords that were present in A. (We note here that representation B includes all possible
redundant parity-check equations there are for the [7,4,3] Hamming code.) Representation $C$ has fewer number of pseudocodewords compared to $A$. However, the set of irreducible pseudocodewords of $C$ is not a subset of the set of irreducible pseudocodewords of $A$. The performance of iterative decoding on representation $C$ indicates a small fraction of detected errors.

Figure 19 compares the performance of min-sum decoding on the three representations. Clearly, $B$, having the best pseudocodeword weight distribution among the three representations, yields the best performance with min-sum iterative decoding, with performance almost matching that of the optimal ML decoder. (Figure 19 shows the bit error rates as solid lines and the frame error rates as dotted lines.)

Similarly, we also analyzed three different representations of the [15,11,3] Hamming code. Representation $A$ has its parity check matrix in the standard systematic form and thus, the corresponding Tanner graph has 4 variable nodes of degree one. Representation $B$ includes all possible redundant parity check equations of representation $A$, meaning the parity check rows of representation $B$ include all linear combinations of the parity check rows of representation $A$, and has the best pseudocodeword-weight distribution. Representation $C$ includes up to order-two redundant parity check equations from the parity check matrix of representation $A$, meaning, the parity check matrix of representation $C$ contained all linear combinations of every pair of rows.
Performance of the [7,4,3] Hamming code with min-sum iterative decoding over the BIAWGNC.

in the parity check matrix of representation $A$. Thus, its (lift-realizable) pseudocodeword-weight distribution
is superior to that of $A$ but inferior to that of $B$. (We have not been able to plot a pseudocodeword-weight
distribution for the different representations of the [15,11,3] Hamming code since the task of enumerating all
irreducible lift-realizable pseudocodewords for these representations proved to be computationally intensive.)

The analogous performance of min-sum iterative decoding of representations $A$, $B$, and $C$ of the [15,11,3]
Hamming code on the binary input (BPSK modulated) additive white Gaussian noise (AWGN) channel with signal to noise ratio (SNR) $E_b/N_0$ is shown in Figures 20, 21, and 22, respectively. (The maximum number of decoding iterations was fixed at 100.) As in the previous example, here also we observe similar trends in the performance curves. $A$ shows a prominent detected error rate, whereas $B$ and $C$ show no presence of detected errors at all. The results indicate that merely adding order two redundant check nodes to the graph of $A$ is sufficient to remove most of the low-weight pseudocodewords.

Figure 23 compares the performance of min-sum decoding on the three representations. Here again, representation $B$, having the best pseudocodeword-weight distribution among the three representations, yields the best performance with min-sum iterative decoding and is closest in performance to that of the ML decoder.

Inferring from the empirical results of this section, we comment that LDPC codes that have structure and redundant check nodes, for example, the class of LDPC codes obtained from finite geometries [8], are likely to have fewer number of low-weight pseudocodewords in comparison to other randomly constructed LDPC graphs of comparable parameters. Despite the presence of a large number of short cycles (i.e., 4-cycles and 6-cycles), the class of LDPC codes in [8] perform very well with iterative decoding. It is worth investigating how the set of pseudocodewords among existing LDPC constructions can be improved, either by adding redundancy or modifying the Tanner graphs, so that the number of (bad) pseudocodewords, both lift-realizable ones as well as those occurring on the computation tree, is lowered.

VIII. EXAMPLES

In this section we present three different examples of Tanner graphs which give rise to different types of pseudocodewords.

Example 1 in Figure 24 shows a graph that has no pseudocodeword with weight less than $d_{\text{min}}$ on the BSC and AWGN channels. For this code (or more precisely, LDPC constraint graph), the minimum distance, the minimum stopping set size, and the minimum pseudocodeword weight on the AWGN channel, are all equal to 4, i.e., $d_{\text{min}} = s_{\text{min}} = w_{\text{min}} = 4$, and the $t$-value (see Section 5) is 2. A non-codeword irreducible
Performance of the [15,11,3] Hamming code with min-sum iterative decoding over the BIAWGNC.

A pseudocodeword with a component of value 2 may be observed by assigning value 1 to the nodes in the outer and inner rings and assigning value 2 to exactly one node in the middle ring, and zeros elsewhere. Figure 25 shows the performance of this code on a binary input additive white Gaussian noise channel, with signal to noise ratio (SNR) $E_b/N_o$, with min-sum, sum-product, and maximum-likelihood decoding. The min-sum and sum-product iterative decoding was performed for 50 decoding iterations on the LDPC constraint graph. It
is evident that all three algorithms perform almost identically. Thus, the LDPC code of Example 1 does not have low weight (relative to the minimum distance) bad pseudocodewords, implying that the performance of the min-sum decoder, under i.i.d. Gaussian noise, will be close to the optimal ML performance.

Example 2 in Figure 26 shows a graph that has both good and bad pseudocodewords. Consider $p = (1, 0, 1, 1, 1, 3, 0, 0, 1, 1, 1, 0)$. Letting $w = (1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0)$, we obtain $pw^T = -2$ and $cw^T \geq 0$ for all codewords $c$. Therefore, $p$ is a bad pseudocodeword for min-sum iterative decoding. In particular, this pseudocodeword has a weight of $w^{BSC/AWGN}(p) = 8$ on both the BSC and the AWGN channels. This LDPC graph results in an LDPC code of minimum distance $d_{\min} = 8$, whereas the minimum stopping set size and minimum pseudocodeword weight (AWGN channel) of the graph are 3, i.e., $s_{\min} = w_{\min} = 3$, and the $t$-value is 8.
Fig. 26. A graph with good and bad pseudocodewords.

Fig. 27. Performance of Example 2 - LDPC code: Min-Sum, Sum-Product, ML decoding over the BIAWGNC.

Figure 27 shows the performance of this code on a binary input additive white Gaussian noise channel with min-sum, sum-product, and maximum-likelihood decoding. The min-sum and sum-product iterative decoding was performed for 50 decoding iterations on the LDPC constraint graph. It is evident that the min-sum and the sum-product decoders are inferior in performance in comparison to the optimal ML decoder. Since the minimal pseudocodeword weight $w_{\text{min}}$ is much less than the minimum distance of the code $d_{\text{min}}$, the performance of the min-sum iterative decoder at high signal to noise ratios (SNRs) is dominated by low-weight bad pseudocodewords.

Example 3 in Figure 28 shows a graph on $m + 1$ variable nodes, where all but the left-most variable node form a minimal stopping set of size $m$, i.e., $s_{\text{min}} = m$. When $m$ is even, the only irreducible pseudocodewords are of the form $(k, 1, 1, \ldots, 1)$, where $0 \leq k \leq m$, and $k$ is even. When $m$ is odd, the irreducible pseudocodewords have the form $(k, 1, 1, \ldots, 1)$, where $1 \leq k \leq m$, and $k$ is odd, or $(0, 2, 2, \ldots, 2)$. In general, any pseudocodeword of this graph is a linear combination of these irreducible pseudocodewords. When $k$ is not 0 or 1, then these are nc-irreducible pseudocodewords; the weight vector $w = (w_1, \ldots, w_{m+1})$, where $w_1 = -1$, $w_2 = +1$, and $w_3 = \ldots = w_{m+1} = 0$, shows that these pseudocodewords are bad. When
Fig. 28. A graph with only bad nc-pseudocodewords.

Fig. 29. Performance of Example 3 - LDPC code for $m = 11$: Min-Sum, Sum-Product, ML decoding over the BIAWGNC.

Fig. 30. Performance of Example 3 - LDPC code for $m = 10$: Min-Sum, Sum-Product, ML decoding over the BIAWGNC.

$m$ is even or odd, any reducible pseudocodeword of this graph that includes at least one nc-irreducible pseudocodeword in its sum, is also bad (according to Definition 2.9). We also observe that for both the BSC and AWGN channels, all of the irreducible pseudocodewords have weight at most $d_{\text{min}} = m$ or $m + 1$, depending on whether $m$ is even or odd. The minimum pseudocodeword weight $w_{\text{min}}^{\text{AWGN}} = 4m/(m + 1)$, and the LDPC constraint graph has a $t$-value of $m$.

Figures 29 and 30 show the performance of the code for odd and even $m$, respectively, on a binary input additive white Gaussian noise channel with min-sum, sum-product, and maximum-likelihood decoding. The
min-sum and sum-product iterative decoding was performed for 50 decoding iterations on the respective LDPC constraint graphs. The performance difference between the min-sum (respectively, the sum-product) decoder and the optimal ML decoder is more pronounced for odd $m$. (In the case of even $m$, $(0, 2, 2 \ldots , 2)$ is not a bad pseudocodeword unlike in the case for odd $m$; thus, one can argue that, relatively, there are a fewer number of bad pseudocodewords when $m$ is even.) Since the graph has low weight bad pseudocodewords, in comparison to the minimum distance, the performance of the min-sum decoder in the high SNR regime is clearly inferior to that of the ML decoder.

Note that the graph in Example 1 has no minimal stopping sets with property $\Theta$, and all stopping sets have size at least $d_{\text{min}}$. However, all graphs have problematic nodes and conditions 1 and 3 in Theorem 4.2 are met in Examples 2 and 3. Specifically, the problematic nodes are the variable nodes in the inner ring of Example 1, $v_7, v_8$ in Example 2, and $v_1$ in Example 3.

This section has demonstrated three particular LDPC constraint graphs having different types of pseudocodewords, leading to different performances with iterative decoding in comparison to optimal decoding. In particular, we observe that the presence of low weight nc-ir-reducible pseudocodewords, with weight relatively smaller than the minimum distance of the code, can adversely affect the performance of iterative decoding.

IX. Conclusions

We provided new lower bounds on the minimum pseudocodeword weight over the BSC and AWGN channels. We examined the structure of pseudocodewords in Tanner graphs and gave sufficient conditions for when pseudocodewords may be problematic with min-sum iterative decoding, and compared the problematic pseudocodeword sets across different channels. We established bounds on the smallest lift degree needed to realize a given pseudocodeword having finite entries. In the case where an upper bound on the maximal component for all irreducible pseudocodewords exists and is known, this bound may be useful in reducing the complexity of determining the minimal pseudocodeword weight. We showed that the graph-covers-polytope definition of [5] in characterizing the set of pseudocodewords of a Tanner graph for min-sum iterative decoding does not capture the entire set of pseudocodewords. We illustrated the effects of different pseudocodewords on iterative decoding and observed that the appropriate addition of redundant check nodes to the graph can
possibly decrease the number of low-weight pseudocodewords. Further, we provided examples of graphs with different kinds of pseudocodewords. Since we dealt primarily with lift-realizable pseudocodewords, these results are also applicable in the analysis of LP decoding. We hope the insights gained from this paper will aid in the design of LDPC codes with good minimum pseudocodeword weights.

APPENDIX

PRELIMINARIES

Lemma 2.1 Proof: Observe that every cloud of check nodes in the corresponding cover of $G$ is connected to either none or at least two of the variable clouds in the support of $p$. If this were not the case, then there would be a cloud of check nodes in the cover with at least one check node in that cloud connected to exactly one variable node of bit value one, thus, leaving the check node constraint unsatisfied. Therefore, the corresponding variable nodes in the base graph $G$ satisfy the conditions of a stopping set.

BOUNDS ON MINIMAL PSEUDOCODEWORD WEIGHTS

Theorem 3.1 Proof:

\[
\alpha_i \leq \sum_{j \neq i} \alpha_j
\]

Single parity check code.

\[
d\alpha_0 \leq \sum_{j \in L_0} \alpha_j, \\
d(d-1)\alpha_0 \leq \sum_{j \in L_1} \alpha_j, \\
\vdots
\]

Local tree structure for a $d$-left regular graph.
Case: $2g$ odd. At a single constraint node, the following inequality holds:

$$\alpha_i \leq \sum_{j \neq i}^{\alpha_j} \sum_{j \in L_0}^{\alpha_j},$$

where $L_0$ corresponds to variable nodes in the first level (level 0) of the tree. Similarly, we have

$$d(d-1)\alpha_0 \leq \sum_{j \in L_1}^{\alpha_j},$$

and so on, until,

$$d(d-1)g\alpha_0 \leq \sum_{j \in L_{g-6}}^{\alpha_j} \sum_{i \in 0 \cup L_0 \cup \ldots L_{g-6}}^{\alpha_i} \sum_{all} \alpha_i.$$

Since the LDPC graph has girth $g$, the variable nodes up to level $L_{g-6}$ are all distinct. The above inequalities yield:

$$[1 + d + d(d-1) + \ldots + d(d-1)^{\frac{g-6}{d}}] \alpha_0 \leq \sum_{i \in 0 \cup L_0 \cup \ldots L_{g-6}}^{\alpha_i} \sum_{all} \alpha_i$$

Without loss of generality, let us assume, $\alpha_0, \ldots, \alpha_{e-1}$ to be the $e$ dominant components in $p$. That is, $\alpha_0 + \alpha_1 + \ldots + \alpha_e \geq \sum_{i=0}^{e-1} \frac{p_i}{2}$. Since, each is at most $\alpha_0$, we have $\sum_{i=0}^{e-1} \alpha_i \leq e\alpha_0$. This implies that

$$e\alpha_0 \geq \sum_{i=0}^{e-1} \alpha_i \geq \sum_{i=0}^{e-1} \frac{\alpha_i}{2} \geq \frac{[1 + d + d(d-1) + \ldots + d(d-1)^{\frac{g-6}{d}}] \alpha_0}{2}$$

$$\Rightarrow e \geq \frac{[1 + d + d(d-1) + \ldots + d(d-1)^{\frac{g-6}{d}}]}{2}.$$

Since $w_{BSC} = 2e$, the result follows. (The case when $\frac{g}{2}$ is even is treated similarly.)

**AWGN case:** Let $x = \frac{[1 + d + d(d-1) + \ldots + d(d-1)^{\frac{g-6}{d}}]}{2}$. Since,

$$\sum_{i=0}^{e-1} \alpha_i \geq \frac{[1 + d + d(d-1) + \ldots + d(d-1)^{\frac{g-6}{d}}] \alpha_0}{2},$$

Note that $L_i$ refers to the level for which the exponent of the $(d-1)$ term is $i$. 

June 27, 2005
we can write \( \sum_{i=0}^{e-1} \alpha_i = (x + y)\alpha_0 \), where \( y \) is some non-negative quantity. Suppose \( \alpha_0 + \ldots + \alpha_{e-1} = \alpha_e + \ldots \alpha_{n-1} \). Then,

\[
w = \left( \frac{\sum_{i=0}^{n-1} \alpha_i^2}{\sum_{i=0}^{n-1} \alpha_i^2} \right)^2 \geq \frac{(2 \sum_{i=0}^{e-1} \alpha_i^2)}{2 \sum_{i=0}^{e-1} \alpha_i^2}
\]

Since we have \( \sum_{i=0}^{n-1} \alpha_i^2 \leq 2 \sum_{i=0}^{e-1} \alpha_i^2 \leq 2\alpha_0 (\sum_{i=0}^{e-1} \alpha_i) = 2(x + y)\alpha_0^2 \), we get,

\[
w \geq \frac{4(x + y)^2\alpha_0^2}{2(x + y)\alpha_0^2} = 2(x + y) \geq 2x.
\]

(The case \( \alpha_0 + \ldots + \alpha_{e-1} > \alpha_e + \ldots \alpha_{n-1} \) is treated similarly.)

**Theorem 3.2** *Proof:* As in the proof of Theorem 3.1, where we note that for a single constraint with neighbors having pseudocodeword components \( \alpha_1, \ldots, \alpha_k \), we have the following relation (for \( \alpha_i, i = 1, \ldots, k \)):

\[(\varepsilon k)\alpha_i \leq \sum_{j=1}^{k} \alpha_j.
\]

The result follows by applying this inequality at every constraint node as in the proof of Theorem 3.1.

**Lemma 3.1** *Proof:* Let \( q \) be odd. Consider the LDPC graph enumerated as a tree from a root node. Without loss of generality, let the root node participate in a stopping set. Then, at the first level of constraint nodes, there must be at least \( \varepsilon k - 1 \) variable nodes connecting each constraint node that the root node is connected to, and all these variable nodes also participate in the same stopping set. Hence there are at least \( d(\varepsilon k - 1) \) other variable nodes at the first variable level from the root node, also belonging to the same stopping set. The constraint nodes that these variable node are connected to in the subsequent level must each have \( \varepsilon k - 1 \) additional variable node neighbors belonging to the stopping set, and so on. Enumerating in this manner up to the \( L_{\varepsilon q} \) level of variable nodes from the root node, where \( L_{\varepsilon q} \) is as in Theorem 3.1, the number of variable nodes belonging to the stopping set that includes the root node is at least \( 1 + d(\varepsilon k - 1) + d(d - 1)(\varepsilon k - 1)^2 + \ldots + d(d - 1)^{\varepsilon q}(\varepsilon k - 1)^{\varepsilon q - 2} \). Since we began with an arbitrary root node, we have \( s_{\min} \geq 1 + d(\varepsilon k - 1) + d(d - 1)(\varepsilon k - 1)^2 + \ldots + d(d - 1)^{\varepsilon q}(\varepsilon k - 1)^{\varepsilon q - 2} \). The case when \( \frac{q}{2} \) is even is treated similarly.
Lemma 3.2 Proof: Let \( p = (\alpha_1, \ldots, \alpha_n) \) be a pseudocodeword of \( G \), and without loss of generality, let \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \). To establish the inequality for the AWGN channel, we need to show

\[
\frac{(\sum_{i=1}^{n} \alpha_i)^2}{\sum_{i=1}^{n} \alpha_i^2} \geq \frac{\sum_{i=1}^{n} \alpha_i}{\max_i \alpha_i} = \frac{\sum_{i=1}^{n} \alpha_i}{\alpha_1}.
\]

But we have \( \sum_i \alpha_i^2 \leq \alpha_1^2 + \alpha_1 \alpha_2 + \ldots + \alpha_1 \alpha_n = \alpha_1 (\sum_i \alpha_i) \). Therefore,

\[
\frac{(\sum_{i=1}^{n} \alpha_i)^2}{\sum_{i=1}^{n} \alpha_i^2} \geq \frac{(\sum_{i=1}^{n} \alpha_i)^2}{\alpha_1 \sum_{i=1}^{n} \alpha_i} \geq \frac{\sum_{i=1}^{n} \alpha_i}{\alpha_1}.
\]

Hence, \( w^{AWGN}(p) \geq w_{\max-\frac{\alpha_1}{\alpha_1}}(p) \).

To establish the bound for the BSC channel, let \( e \) be the smallest number such that \( \sum_{i=1}^{e} \alpha_i \geq \frac{\sum_{i=1}^{n} \alpha_i}{2} \).

First suppose \( \sum_{i=1}^{e} \alpha_i = \sum_{i=e+1}^{n} \alpha_i \). Then \( w^{BSC}(p) = 2e \). Moreover, \( w_{\max-\frac{\alpha_1}{\alpha_1}}(p) = \frac{\sum_{i=1}^{e} \alpha_i}{\alpha_1} = 2 \frac{\sum_{i=1}^{e} \alpha_i}{\alpha_1} \).

Each \( \alpha_i \leq \alpha_1 \Rightarrow w_{\max-\frac{\alpha_1}{\alpha_1}}(p) \leq \frac{2e \alpha_1}{\alpha_1} = 2e = w^{BSC}(p) \). Now suppose \( \sum_{i=1}^{e} \alpha_i > \sum_{i=e+1}^{n} \alpha_i \). Then, for some \( \delta > 0 \), we have \( \sum_{i=1}^{e} \alpha_i = \sum_{i=e+1}^{n} \alpha_i + \delta \). We have \( w_{\max-\frac{\alpha_1}{\alpha_1}}(p) = \frac{\sum_{i=1}^{e} \alpha_i}{\alpha_1} = \frac{\sum_{i=1}^{e} \alpha_i + \sum_{i=e+1}^{n} \alpha_i}{\alpha_1} \).

Note that \( \sum_{i=1}^{e} \alpha_i + \sum_{i=e+1}^{n} \alpha_i < 2 \sum_{i=1}^{e} \alpha_i < (2e)\alpha_1 \). Thus, \( w_{\max-\frac{\alpha_1}{\alpha_1}}(p) < \frac{2e \alpha_1}{\alpha_1} = 2e \).

Lemma 3.3 Proof: (a) AWGN case: Let \( n_k \) be the number of \( p_i \)'s that are equal to \( k \), for \( k = 1, \ldots, t \).

The pseudocodeword weight is then equal to:

\[
w^{AWGN}(p) = \frac{(n_1 + 2n_2 + \ldots + tn_t)^2}{(n_1 + 2^2n_2 + \ldots + t^2n_t)}.
\]

Now, we have to find a number \( r \) such that \( w^{AWGN}(p) \geq r |\text{supp}(p)| \). Note however, that \( |\text{supp}(p)| = n_1 + n_2 + \ldots + n_t \). This implies that for an appropriate choice of \( r \), we have

\[
\frac{(n_1 + 2n_2 + \ldots + tn_t)^2}{(n_1 + 2^2n_2 + \ldots + t^2n_t)} \geq r(n_1 + \ldots + n_t) \quad \text{or} \quad \sum_{i=1}^{t} i^2n_i^2 \geq \sum_{i=1}^{t} \sum_{j=i+1}^{t} \frac{(i^2 + j^2)r - 2ij}{1 - r}n_in_j
\]

Note that \( r < 1 \) in the above. Clearly, if we set \( r \) to be the minimum over all \( 1 \leq i < j \leq n \) such that \( (i^2 + j^2)r \leq 2ij \), then it can be verified that this choice of \( r \) will ensure that (*) is true. This implies \( r = \frac{2i}{i+j} \) (for \( i = 1, j = t \)).

However, observe that left-hand-side (LHS) in (*) can be written as the following LHS:

\[
\frac{1}{t-1} \sum_{i=1}^{t} \sum_{j=i+1}^{t} (j^2n_i^2 + j^2n_j^2) \geq \sum_{i=1}^{t} \sum_{j=i+1}^{t} \frac{(i^2 + j^2)r - 2ij}{1 - r}n_in_j
\]
Now, using the inequality $a^2 + b^2 \geq 2ab$, $r$ can be taken as the minimum over all $1 \leq i < j \leq n$ such that
\[
\frac{1}{t^2}(i^2 n_i^2 + j^2 n_j^2) \geq \frac{r(i^2+j^2-2i)}{1+r^2}. \]
This gives $r = \frac{2i^2}{(1+r^2)(t-1)+2t}$, thereby proving the lemma in the AWGN case.

**BSC case:** Let
\[
\begin{align*}
tn_t + (t-1)n_{t-1} + \ldots + jn_j + a(j-1) &\geq (n_{j-1} - a)(j-1) + (j-2)n_{j-2} + \ldots + n_1, \\
\end{align*}
\]
where $0 \leq a \leq n_{j-1}$ is the smallest number for some $1 \leq j \leq t$ such that the above inequality holds. **Case 1:** If
\[
\begin{align*}
tn_t + (t-1)n_{t-1} + \ldots + jn_j + a(j-1) &\geq (n_{j-1} - a)(j-1) + (j-2)n_{j-2} + \ldots + n_1, \\
\end{align*}
\]
then, $w_{BSC}(p) = 2(n_t + n_{t-1} + \ldots + n_j + a)$. But observe,
\[
\begin{align*}
w_{BSC}(p) &= 2(n_t+n_{t-1}+\ldots+n_j+a) = \frac{2t(n_t+n_{t-1}+\ldots+n_j+a)}{t} \geq \frac{1}{t}2(tn_t+(t-1)n_{t-1}+\ldots+jn_j+a(j-1)) \\
&= \frac{1}{t}(tn_t+(t-1)n_{t-1}+\ldots+(j-1)a+(j-1)(n_{j-1}-a)+\ldots+n_1) \geq \frac{1}{t}(n_1+n_2+\ldots+n_t) = \frac{1}{t}|\text{supp}(p)|
\end{align*}
\]

**Case 2:** If
\[
\begin{align*}
tn_t + (t-1)n_{t-1} + \ldots + jn_j + a(j-1) &> (n_{j-1} - a)(j-1) + (j-2)n_{j-2} + \ldots + n_1, \\
\end{align*}
\]
and
\[
\begin{align*}
tn_t + (t-1)n_{t-1} + \ldots + jn_j + (a-1)(j-1) &< (n_{j-1} - a + 1)(j-1) + (j-2)n_{j-2} + \ldots + n_1, \\
\end{align*}
\]
then, $w_{BSC}(p) = 2(n_t + n_{t-1} + \ldots + n_j + a) - 1$. But observe,
\[
\begin{align*}
w_{BSC}(p) &= 2(n_t+n_{t-1}+\ldots+n_j+a)-1 = \frac{2t(n_t+n_{t-1}+\ldots+n_j+a)}{t} - 1 \geq \frac{1}{t}2(tn_t+(t-1)n_{t-1}+\ldots+jn_j+a(j-1))-1 \\
&= \frac{1}{t}(tn_t+(t-1)n_{t-1}+\ldots+(j-1)a+(j-1)(n_{j-1}-a)+\ldots+n_1)-1 \geq \frac{1}{t}(n_1+n_2+\ldots+n_t)-1 = \frac{1}{t}|\text{supp}(p)|-1
\end{align*}
\]
Therefore, $w_{BSC}(p) \geq \frac{1}{t}|\text{supp}(p)|$.

**Theorem 3.5** **Proof:** Let $p$ be a good pseudocodeword. This means that if for any weight vector $w$ we have $cw^T \geq 0$ for all $0 \neq c \in C$, then, $pw^T \geq 0$. Let us now consider the BSC and the AWGN cases separately.
BSC case: Suppose at most \( \frac{d_{\text{min}}}{2} \) errors occur in channel. Then, the corresponding weight vector \( \mathbf{w} \) will have \( \frac{d_{\text{min}}}{2} \) or fewer \(-1\) components and rest \(+1\) components. This implies that the cost of any \( \mathbf{0} \neq \mathbf{c} \in C \) (i.e., \( \mathbf{c}^T \mathbf{w} \)) is at least 0 since there are at least \( d_{\text{min}} \) 1’s in support of any \( \mathbf{0} \neq \mathbf{c} \in C \). Since \( \mathbf{p} \) is a good pseudocodeword, it must also have positive cost, \( \mathbf{p}^T \mathbf{w} \geq 0 \). Let us assume that the \(-1\)’s occur in the dominant \( \frac{d_{\text{min}}}{2} \) positions of \( \mathbf{p} \), and without loss of generality, assume \( p_1 \geq p_2 \geq \ldots \geq p_n \). (Therefore, \( \mathbf{w} = (-1, -1, \ldots, -1, +1, +1, \ldots, +1) \).) Positive cost of \( \mathbf{p} \) implies \( p_1 + \ldots + p_{\frac{d_{\text{min}}}{2}} \leq p_{\frac{d_{\text{min}}}{2} + 1} + \ldots + p_n \). So we have \( e \geq \frac{d_{\text{min}}}{2} \), where \( e \) is as defined in the pseudocodeword weight of \( \mathbf{p} \) for the BSC channel. The result follows.

AWGN case: Without loss of generality, let \( p_1 \) be dominant component of \( \mathbf{p} \). Set the weight vector \( \mathbf{w} = (1 - d_{\text{min}}, 1, \ldots, 1) \). Then it can be verified that \( \mathbf{c}^T \mathbf{w} \geq 0 \) for any \( \mathbf{0} \neq \mathbf{c} \in C \). Since \( \mathbf{p} \) is a good pseudocodeword, this implies \( \mathbf{p} \) also must have positive cost. Cost of \( \mathbf{p} \) is \( (1 - d_{\text{min}})p_1 + p_2 + \ldots + p_n \geq 0 \Rightarrow \frac{p_1 + \ldots + p_n}{p_1} \geq \frac{d_{\text{min}}}{2} \). Note, that the right-hand-side (RHS) is \( w_{\text{max}} - \text{frac}(\mathbf{p}) \); hence, the result follows from Lemma 3.2.

Now let us consider \( \mathbf{p} \) to be a bad pseudocodeword. From Lemma 2.1, we have \( |\text{supp}(\mathbf{p})| \geq s_{\text{min}} \). Therefore, \( w_{\text{max}} - \text{frac}(\mathbf{p}) \geq \frac{s_{\text{min}}}{t} \) (since \( p_1 = t \) is the maximum component of \( \mathbf{p} \)), and hence, the result follows by Lemmas 3.2 and 3.3.

**Structure of Pseudocodewords**

**Lemma 4.1 Proof:** Consider a graph \( H \) having a single check node which is connected to variable nodes \( v_1, \ldots, v_k \). Suppose \( \mathbf{b} = (b_1, \ldots, b_k) \) is a pseudocodeword in \( H \), then \( \mathbf{b} \) corresponds to a codeword in a lift \( \hat{H} \) of \( H \). Every check node in \( \hat{H} \) is connected to an even number of variable nodes that are assigned value 1, and further, each variable node is connected to exactly one check node in the check cloud. Since the number of variable nodes that are assigned value 1 is equal to the sum of the \( b_i \)'s, we have \( \sum_i b_i \equiv 0 \mod 2 \).

Let \( \hat{G} \) be the corresponding lift of \( G \) wherein \( \mathbf{p} \) forms a valid codeword. Then each check node in \( \hat{G} \) is connected to an even number of variable nodes that are assigned value 1. From the above observation, if nodes \( v_{i_1}, \ldots, v_{i_k} \) participate in the check node \( c_i \) in \( G \), then \( p_{i_1} + \ldots + p_{i_k} \equiv 0 \mod 2 \). Let \( x_i = p_i \mod 2 \), for \( i = 1, \ldots, n \) (\( n \) being the number of variable nodes, i.e., the block length of \( C \), in \( G \)). Then, at every check
node $c_i$, we have $x_{i_1} + \ldots + x_{i_k} \equiv 0 \pmod{2}$. Since $x = (x_1, \ldots, x_n) = p \pmod{2}$ is a binary vector satisfying all checks, it is a codeword in $C$.

**Lemma 4.2**  
*Proof:* Suppose $c \in C$ is in the support of $p$, then form $p' = p - c$. If $p'$ contains a codeword in its support, then repeat the above step on $p'$. Subtracting codewords from the pseudocodeword vector in this manner will lead to a decomposition of the vector $p$ as stated. Observe that the residual vector $r$ contains no codeword in its support.

From Lemma 4.1, $x = p \pmod{2}$ is a codeword in $C$. Since $p = c_1 + \ldots + c_k + r$, we have $x = (c_1 + \ldots + c_k) \pmod{2} + r \pmod{2}$. But since $x \in C$, this implies $r \pmod{2} \in C$. However, since $r$ contains no codeword in its support, $r \pmod{2}$ must be the all-zero codeword. Thus, $r$ contains only even (possibly 0) components.

**Theorem 4.1**  
*Proof:* Let $p$ be a pseudocodeword of a code $C$, and suppose $p$ may be decomposed as $p = c_1 + c_2 + \ldots + c_k$, where $\{c_i\}_{i=1}^k$ is a set of not necessarily distinct codewords. Suppose $p$ is bad. Then there is a weight vector $w$ such that $pw^T < 0$ but for all codewords $c \in C$, $cw^T \geq 0$. Having $pw^T < 0$ implies that $c_1w^T + c_2w^T + \ldots + c_kw^T = -x$, for some positive real value $x$. So there is at least one $i$ for which $c_iw^T < 0$, which is a contradiction. Therefore, $p$ is a good pseudocodeword.

**Theorem 4.2**  
*Proof:*  
1) If $w_{BSC/AWGN}^{BSC/AWGN}(p) < d_{\min}$, then $p$ is a bad pseudocodeword by Theorem 3.5.

2) If $|\text{supp}(p)| < d_{\min}$, then there is no codeword in the support of $p$, by Lemma 4.1. Let $w = (w_1, \ldots, w_n)$, be a weight vector where for $i = 1, 2, \ldots, n$,

\[
  w_i = \begin{cases} 
    -1 & \text{if } v_i \in \text{supp}(p), \\
    +\infty & \text{if } v_i \notin \text{supp}(p) 
  \end{cases}
\]

Then $pw^T < 0$ and for all codewords $c \in C$, $cw^T \geq 0$.

3) Suppose $p$ is a non-codeword irreducible pseudocodeword. Without loss of generality, assume $p = (p_1, p_2, \ldots, p_s, 0, 0, \ldots, 0)$, i.e., the first $s$ positions of $p$ are non-zero and the rest are zero. Suppose $p$ contains $\ell$ distinct codewords $c^{(1)}, c^{(2)}, \ldots, c^{(\ell)}$ in its support. Then if $s \geq \ell + 1$, we define a weight vector $w = (w_1, w_2, \ldots, w_n)$ as follows. Let $w_i = +\infty$ for $i \notin \text{supp}(p)$. Solve for $w_1, w_2, \ldots, w_s$
from the following system of linear equations:

\[
\begin{align*}
\mathbf{c}_1^T \mathbf{w} &= +1, \\
\mathbf{c}_2^T \mathbf{w} &= +1, \\
&\vdots \\
\mathbf{c}_{\ell}^T \mathbf{w} &= +1, \\
\mathbf{p}^T \mathbf{w} &= p_1 w_1 + p_2 w_2 + \ldots + p_s w_s = -2,
\end{align*}
\]

The above system of equations involve \(s\) unknowns \(w_1, w_2, \ldots, w_s\) and there are \(\ell + 1\) equations. Hence, as long as \(s \geq \ell + 1\), we have a solution for the \(w_i\)'s. Thus, there exists a weight vector \(\mathbf{w} = (w_1, \ldots, w_s, +\infty, \ldots, +\infty)\) such that \(\mathbf{p}^T \mathbf{w} < 0\) and \(\mathbf{c}^T \mathbf{w} \geq 0\) for all codewords \(\mathbf{c}\) in the code. This proves that \(\mathbf{p}\) is a bad pseudocodeword.

\[\blacksquare\]

**Theorem 4.3 Proof:**

1) Let \(S\) be a stopping set. **Suppose there are no non-zero codewords whose support is contained in \(S\).** The pseudocodeword \(\mathbf{p}\) with component value 2 in the positions of \(S\), and 0 elsewhere is then a bad pseudocodeword on the AWGN channel, which may be seen by the weight vector \(\mathbf{w}_a = (w_1, \ldots, w_n)\), where for \(i = 1, 2, \ldots, n,\)

\[
w_i = \begin{cases} 
-1 & \text{if } i \in S, \\
+\infty & \text{if } i \notin S
\end{cases}
\]

In addition, since all nonzero components have the same value, the weight of \(\mathbf{p}\) on the BSC and AWGN channels is \(|S|\).

Suppose now that \(\mathbf{p}\) is a non-codeword pseudocodeword with support \(S\). Then the weight vector \(\mathbf{w}_a\) again shows that \(\mathbf{p}\) is bad, i.e., \(\mathbf{p}^T \mathbf{w}_a < 0\) and \(\mathbf{c'}^T \mathbf{w}_a \geq 0\) for all \(\mathbf{c'} \in \mathcal{C}\).

2) **Suppose there is at least one non-zero codeword \(\mathbf{c}\) whose support is in \(S\).**

   (a) **Assume \(S\) is a minimal stopping set.** Then this means that \(\mathbf{c}\) is the only non-zero codeword whose support is in \(S\) and \(\text{supp}(\mathbf{c}) = S\).
(i) Suppose $S$ has property $\Theta$, then we can divide the variable nodes in $S$ into disjoint equivalence classes such that the nodes belonging to each class are connected pairwise by a path traversing only via degree two check nodes in $G_{|S}$. Consider the pseudocodeword $p$ having component value 3 in the positions corresponding to all nodes of one equivalence class, component value 1 for the remaining positions of $S$, and component value 0 elsewhere. Let $r = p - c$, and let $\hat{i}$ denote the index of the first non-zero component of $r$ and $i^*$ denote the index of the first non-zero component in $\text{supp}(p) - \text{supp}(r)$. The weight vector $w_b = (w_1, \ldots, w_n)$, where for $i = 1, 2, \ldots, n$,

$$
\begin{align*}
  w_i &= \begin{cases} 
    -1 & \text{if } i = \hat{i}, \\
    +1 & \text{if } i = i^*, \\
    +\infty & \text{if } i \notin S, \\
    0 & \text{otherwise}
  \end{cases}
\end{align*}
$$

ensures that $p$ is bad as in Definition 2.9, and it is easy to show that the weight of $p$ on the AWGN channel is strictly less than $|S|$. Conversely, suppose $S$ does not have property $\Theta$. Then every pair of variable nodes in $S$ is connected by a path in $G_{|S}$ that contains only degree two check nodes. This means that any pseudocodeword $p$ with support $S$ must have all its components in $S$ of the same value. Therefore, the only pseudocodewords with support $S$ that arise have the form $p = kc$, for some positive integer $k$. (By Theorem 4.1, these are good pseudocodewords.) Hence, there exists no bad pseudocodewords with support $S$.

(ii) Let $p$ be a non-codeword pseudocodeword with support $S$. If $S$ contains a codeword $c$ in its support, then since $S$ is minimal $\text{supp}(c) = S$. Let $k$ denote the number of times $c$ occurs in the decomposition (as in Lemma 4.2) of $p$. That is, $p = kc + r$. Note that $r$ is non-zero since $p$ is a non-codeword pseudocodeword. Let $\hat{i}$ denote an index of the maximal component of $r$, and let $i^*$ denote the index of the first nonzero component in $\text{supp}(p) - \text{supp}(r)$. The weight
vector $w_b$, defined above, again ensures that $p$ is bad.

(b) **Suppose $S$ is not a minimal stopping set and there is at least one non-zero codeword $c$ whose support is in $S$.**

(i) Suppose $S$ contains a problematic node $v$. By definition of problematic node, let us assume that $S$ is only the stopping set among all stopping sets in $S$ that contains $v$. Then define a set $S_v$ as

$$S_v := \{ u \in S \mid \text{there is a path from } u \text{ to } v \text{ containing only degree two check nodes in } G \}$$

Then, the pseudocodeword $p$ that has component value 4 on all nodes in $S_v$, component value 2 on all nodes in $S - S_v$ and component value 0 everywhere else is a valid pseudocodeword. Let $i^*$ be the index of the variable node $v$ in $G$, and let $i'$ be the index of some variable node in $S - S_v$. Then, the weight vector $w_c = (w_1, \ldots, w_n)$, where for $i = 1, 2, \ldots, n$,

$$w_i = \begin{cases} -1 & \text{if } i = i^*, \\ +1 & i = i' \\ +\infty & i \notin S \\ 0 & \text{otherwise} \end{cases}$$

ensures that $p w_c^T < 0$ and $c' w_c^T \geq 0$ for all non-zero codewords $c'$ in $C$. Hence, $p$ is a bad pseudocodeword with support $S$. This shows the existence of a bad pseudocodeword on $S$.

Suppose now that $p'$ is some non-codeword irreducible pseudocodeword with support $S$. If there is a non-zero codeword $c$ such that $c$ has support in $S$ and contains $v$ in its support, then since $v$ is a problematic node in $S$, $v$ cannot lie in a smaller stopping set in $S$. This means that support of $c$ is equal to $S$. We will show that $p'$ is a bad pseudocodeword by constructing a suitable weight vector $w_d$. Let $i'$ be the index of some variable node in $S - S_v$. Then we can
define a weight vector $w_d = (w_1, \ldots, w_n)$, where for $i = 1, 2, \ldots, n$,

$$w_i = \begin{cases} 
-1 & \text{if } i = i^*, \\
+1 & i = i' \\
+\infty & i \notin S \\
0 & \text{otherwise}
\end{cases}$$

(Note that since $p'$ is a non-codeword irreducible pseudocodeword and contains $v$ in its support, $p'_l > p'_{l'}.)$ This weight vector ensures that $p'w_d^T < 0$ and $c'w_d^T \geq 0$ for all non-zero codewords $c' \in C$. Thus, $p'$ is a bad pseudocodeword.

If there is no non-zero codeword $c$ such that $c$ has support in $S$ and also contains $v$ in its support. Then, the weight vector $w_e = (w_1, \ldots, w_n)$, where for $i = 1, 2, \ldots, n$,

$$w_i = \begin{cases} 
-1 & \text{if } i = i^*, \\
+\infty & i \notin S \\
0 & \text{otherwise}
\end{cases}$$

ensures that $p'w_e^T < 0$ and $c'w_e^T \geq 0$ for all non-zero codewords $c' \in C$. Thus, $p'$ is a bad pseudocodeword.

This proves that any non-codeword irreducible pseudocodeword with support $S$ is bad.

(ii) Suppose $S$ is not a minimal stopping set and suppose $S$ does not contain any problematic nodes. Then, any node in $S$ belongs to a smaller stopping set within $S$. We claim that each node within $S$ belongs to a minimal stopping set within $S$. Otherwise, a node belongs to a proper non-minimal stopping set $S' \subset S$ and is not contained in any smaller stopping set within $S'$ – thereby, implying that the node is a problematic node. Therefore, all nodes in $S$ are contained in minimal stopping sets within $S$.

To prove the last part of the theorem, suppose one of these minimal stopping sets, say $S_j$, is not the support of any non-zero codeword in $C$. Then, there exists a bad non-codeword pseudocodeword $p = (p_1, \ldots, p_n)$ with support $S$, where $p_i = x$, for an appropriately chosen positive even integer $x$ that is at least 4, for $i \in S_j$ and $p_i = 2$ for $i \in S - S_j$, and $p_i = 0$ for
Let $i^*$ be the index of a variable node $v^*$ in $S_j$. If there are distinct codewords $c_1, c_2, \ldots, c_t$ whose supports contains $v^*$ and whose supports are contained in $S$, then let $i_1, i_2, \ldots, i_t'$ be the indices of variable nodes in the supports of these codewords outside of $S_j$. (Note that we choose the smallest number $t' \leq t$ of indices such that each codeword contains one of the variable nodes $v_{i_1}, \ldots, v_{i_{t'}}$ in its support.) The following weight vector $w = (w_1, \ldots, w_n)$ ensures that $p$ is a bad pseudocodeword.

$$w_i = \begin{cases} 
-1 & \text{if } i = i^* \\
+1 & \text{if } i = i_1, i_2, \ldots, i_{t'} \\
+\infty & \text{if } i \not\in S \\
0 & \text{otherwise}
\end{cases}$$

(Note that $x$ can be chosen so that $pw^T < 0$ and it is clear that $cw^T \geq 0$ for all codewords in the code.)

Now suppose $S_j$ contains the support of a codeword $c'$ and has property $\Theta$, then we can construct a bad pseudocodeword on $S_j$ using the previous argument and that in part 2(a)(i) above (since $S_j$ is minimal) and allow the remaining components of $p$ in $S - S_j$ to have a component value of 2. It is easy to verify that such a pseudocodeword is bad.

Conversely, suppose every minimal stopping set $S_j$ within $S$ does not have property $\Theta$ and contains the support of some non-zero codeword $c_j$ within it. Then, this means that $S_j = supp(c_j)$ and that between every pair of nodes within $S_j$ there is a path that contains only degree two check nodes in $G|_{S_j}$. Then, for any pseudocodeword $p$ with support $S$, there is a decomposition of $p$, as in Lemma 4.2, such that $p$ can be expressed as a linear combination of codewords $c_j$'s. Therefore, by Theorem 4.1, there are no bad pseudocodewords with support $S$.

\begin{flushright}
\textbf{Theorem 4.4} \text{ Proof:} \end{flushright}

Let the LDPC code $\mathcal{C}$ represented by the LDPC constraint graph $G$ have blocklength $n$. Note that the weight vector $w = (w_1, w_2, \ldots, w_n)$ has only $+1$ or $-1$ components on the
BSC channel, whereas, it has every component $w_i$ in the interval $[-L, +L]$ on the truncated AWGN channel $T_{AWGN}(L)$, and has every component $w_i$ in the interval $[-\infty, +\infty]$ on the AWGN channel. That is,

$$P_{BSC}^B(G) = \{p | \exists w \in \{+1, -1\}^n \text{ s.t. } pw^T < 0, cw^T \geq 0, \forall 0 \neq c \in C\}.$$ 

$$P_{T_{AWGN}}^B(L)(G) = \{p | \exists w \in [+L, -L]^n \text{ s.t. } pw^T < 0, cw^T \geq 0, \forall 0 \neq c \in C\}.$$ 

$$P_{AWGN}^B(G) = \{p | \exists w \in [+\infty, -\infty]^n \text{ s.t. } pw^T < 0, cw^T \geq 0, \forall 0 \neq c \in C\}.$$ 

It is clear from the above that, for $L \geq 1$, $P_{BSC}^B(G) \subseteq P_{T_{AWGN}}^B(L)(G) \subseteq P_{AWGN}^B(G)$.

Pseudocodewords and Lift-Degrees

**Theorem 5.1**  **Proof:** Let $m$ be the minimum degree lift needed to realize the given pseudocodeword $p$. Then, in a degree $m$ lift graph $\hat{G}$ that realizes $p$, the maximum number of active check nodes in any check cloud is at most $m$. A check cloud $c$ is connected to $\sum_{i \in N(c)} p_i$ active variable nodes from the variable clouds adjoining check cloud $c$. (Note that $N(c)$ represents all the variable clouds adjoining $c$.) Since every active check node in any check cloud has at least two (an even number) active variable nodes connected to it, we have that $2m \leq \max_c \sum_{i \in N(c)} p_i$. This quantity can be upper-bounded by $td_i^+ + r$ since $p_i \leq t$, for all $i$, and $|N(c)| \leq d_i^+$, for all $c$.

**Theorem 5.2**  **Proof:**

1) Suppose $S$ is minimal and does not have property $\Theta$. Then each pair of nodes in $S$ are connected by a path via degree two check nodes, and hence all components of a pseudocodeword $p$ on $S$ are equal. If $S$ is the support of a codeword $c$, then any pseudocodeword is a multiple of $c$, and so $t_S = 1$. If $S$ is not the support of a codeword, then the all-twos vector is the only irreducible pseudocodeword, and $t_S = 2$.

2) Suppose $S$ is minimal and has property $\Theta$. Then we may divide the nodes in $S$ into disjoint equivalence classes so that each pair of nodes within each class is connected by a path via degree two check nodes. Label the classes $S_1, S_2, \ldots, S_k$. Note that if class $i$ connects to class $j$ in $S$, then there is at least
one node in each class that connects with exactly one edge to a check node that has more than one connection to other classes. Such a connection is illustrated in Figure 31.

Fig. 31. Connection between equivalence classes of a minimal stopping set.

Now consider the case where for $i = 1, \ldots, k - 1$, $S_i$ only connects to $S_{i+1}$, as illustrated in Figure 32. Let $\rho_{i,j}^\ell$ be the $\ell^{th}$ check node that interconnects sets $S_i$ and $S_j$ wherein the check node has a degree one connection from $S_i$.

Fig. 32. A serially connected minimal stopping set with property $\Theta$.

Since any pseudocodeword $p$ on $S$ has the same component value for all nodes within an equivalence class, let an arbitrary pseudocodeword $p$ on $S$ have component values $t_1$ for all nodes in $S_1$, $t_2$ for all nodes in $S_2$, and so on. Then, since $p$ is lift-realizable, we have

\[
\frac{t_1}{\max_\ell (\rho_{1,2}^{(\ell)} - 1)} \leq t_2 \leq t_1 (\min_\ell (\rho_{2,1}^{(\ell)} - 1))
\]

\[
\frac{t_2}{\max_\ell (\rho_{2,3}^{(\ell)} - 1)} \leq t_3 \leq t_2 (\min_\ell (\rho_{3,2}^{(\ell)} - 1))
\]

\[
\vdots
\]

\[
\frac{t_{i-1}}{\max_\ell (\rho_{i-1,1}^{(\ell)} - 1)} \leq t_i \leq t_{i-1} (\min_\ell (\rho_{1,i-1}^{(\ell)} - 1)), \quad \text{for } i = 2, \ldots, k
\]
Thus, if the maximum value of any irreducible pseudocodeword in $S_1$ is $t_{s_1}$, then the maximum value for any irreducible pseudocodeword on $S$ is upper bounded by

$$t_{s_k} \leq t_{s_1} \prod_{i=2}^{k} (\min(\rho_{j,i-1}^{(\ell)} - 1))$$

Claim 1.1: $t_{s_1}$ is 1 or 2.

To sketch the proof of claim, we consider two cases. If there is a codeword $c$ whose support is contained in $S$, then by the minimality of $S$, $\text{supp}(c) = S$. In this case, we can construct the following irreducible pseudocodewords. The all-ones vector is an irreducible pseudocodeword. A pseudocodeword that has $t_{s_1} = 1$, $t_{s_2} \leq t_{s_1}(\min(\rho_{j,1}^{(\ell)} - 1))$, ... , is irreducible, for all allowable integer values of $t_2, t_3, \ldots, t_k$ that also satisfy the constraints in (3). We claim that any pseudocodeword $p$ that has a component value larger than $t_{s_k}$ is reducible in terms of the above irreducible pseudocodewords. To prove this claim, assume that the pseudocodeword $p$ has a component value of $t_{s_k} + y$, for a positive integer $y$, for all nodes in class $S_i$. Then this means that $t_{j-1} \geq t_j/(\min(\rho_{j,j-1}^{(\ell)} - 1))$ for $j = i, i-1, i-2, \ldots, 2$, and $t_{j+1} \geq t_j/(\max(\rho_{j,j+1}^{(\ell)} - 1))$ for $j = i, i+1, \ldots, k-1$. Thus, the pseudocodeword will necessarily have component values larger than $t_{S_j}$ for all the sets $S_1, S_2, \ldots, S_k$. Hence, $p$ can be reduced in terms of the irreducible pseudocodewords listed above, where each irreducible pseudocodeword has a component value $t_1 = 1$ for nodes in $S_1$. (If there is no codeword whose support is in $S$, then the same argument as above holds for $t_1 = 2$.)

Therefore, any irreducible pseudocodeword on $S$ has a maximum component value that is upper bounded by $t_{s_k}$.

Now, let us consider a minimal stopping set where the equivalence classes in $S$ are interconnected in an arbitrary manner. We start with an arbitrary class, say $S_1$, and connect it in parallel to the classes to which it connects. Continuing in this way we obtain a general case as in Figure 33. In the general case shown in Figure 33, we may assume that the class $S_1$ at layer $T_0$ interconnects with classes $S_2, S_3, \ldots$, that are represented by layer $T_1$. The classes in layer $T_1$ may interconnect among themselves and to some other classes in layer $T_2$, and so on.

We sketch the proof for the general case as follows: In the general connection scheme there are
more inequality constraints than (3) imposed by the interconnection of classes as described. Hence, the irreducible pseudocodewords with support $S$ will have less flexibility to assume larger component values. Therefore, a similar argument as in the proof of the serial connection scheme holds here. That is, we bound the maximum component value of the classes in layer $T_1$, followed by the classes in layer $T_2$, and so on. Hence, as in the serial case, there is an upper bound on the maximum component value of any irreducible pseudocodeword on $S$ in the general case.

The above claims prove that any irreducible pseudocodeword on $S$ has a maximum component that is bounded by a value $t_S < \infty$.

\section*{Graph-Covers-Polytope Approximation}

\textbf{Claim 6.1} \textit{Proof:} Let $G$ represent the constraint graph of an LDPC code $C$. Suppose $G$ is a tree, then clearly, any pseudocodeword of $G$ can be expressed as a linear combination of codewords of $G$. Hence, suppose $G$ is not a tree, and suppose all check nodes in $G$ are of degree two. Then the computation tree contains only check nodes of degree two and hence, for a valid assignment on the computation tree, the value of any child variable node $v_1$ on the computation tree that stems from a parent check node $h$ is the same as the value of the variable node $v_2$ which is the parent node of $h$. Thus, the only local codeword configurations at each check node is the all-ones configurations when the root node of the tree is assigned the value one. Hence, the only valid solutions on the computation tree correspond to the all ones vector and the all zeros

Fig. 33. A minimal stopping set with an arbitrary connection topology.
vector– which are valid codewords in $C$.

Conversely, suppose $G$ is not a tree and suppose there is a check node $h$ of degree $k$ in $G$. Let $v_1, v_2, \ldots, v_k$ be the variable node neighbors to $h$. Then, enumerate the computation tree rooted at $v_1$ for a sufficient depth such that the node $h$ appears several times in the tree and also as a node in the final check node layer of the tree. Then, there are several possible valid assignments in the computation tree, where the values assigned to the leaf nodes that stem from $h$ yield a solution that is not a valid codeword in $G$. Thus, $G$ contains non-codeword irreducible pseudocodewords on its computation tree.

**Claim 6.2**  
**Proof:** Let $G$ represent the constraint graph of an LDPC code $C$. Suppose $G$ is a tree, then clearly, any pseudocodeword of $G$ can be expressed as a linear combination of codewords of $G$. Hence, suppose $G$ is not a tree, and between every pair of variable nodes in $G$ there is a path that contains only degree two check nodes in $G$. Then $G$ contains only lift-realizable pseudocodewords of the form $(k, k, \ldots, k)$, where $k$ is a positive integer. Note that the all-ones-vector is a valid codeword in $G$. Hence, the only irreducible pseudocodewords in $G$ are the all-zero vector $(0, 0, \ldots, 0)$ and the all-ones vector $(1, 1, \ldots, 1)$ – both being codewords in $G$.  

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**References**


